# CHAPTER 7 <br> Arbitrary Lagrangian Eulerian Formulations 

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### 7.1 Introduction

In Chapter 3, the classical Lagrangian and Eulerian approaches to the description of motion in continuum mechanics were presented. In the Lagrangian approach, the independent variables are taken to be the initial position, $\mathbf{X}$, of a material point and time, $t$. Thus the motion is given by

$$
\begin{equation*}
\mathbf{x}=\mathbf{f}(\mathbf{X}, t)^{\prime} \tag{7.1.1}
\end{equation*}
$$

In this expression, the quantity $\mathbf{x}$ is the position occupied at time $t$ by the material point which occupied the position $\mathbf{X}$ at time $t=0$. The quantity $\mathbf{f}$ is a mapping which describes the motion in terms of the independent variables $\mathbf{X}$ and $t$, and $\mathbf{x}$ is the value of the mapping for the values $\mathbf{X}$ and $t$. Recall that, in the Lagrangian description, the distinction between the value $\mathbf{x}$ and the mapping $\mathbf{f}$ is often ignored and we write $\mathbf{x}=\mathbf{x}(\mathbf{X}, t)$. A scalar field $F$, for example, may be represented by

$$
\begin{equation*}
F=F(\mathbf{X}, t) \tag{7.1.2}
\end{equation*}
$$

In the Eulerian description, the independent variables are spatial position $\mathbf{x}$ and time $t$. A scalar field in the Eulerian description may then be given by

$$
\begin{equation*}
f=f(\mathbf{x}, t) \tag{7.1.4}
\end{equation*}
$$

The field can be represented in terms of either the Eulerian or Lagrangian coordinates as follows

$$
\begin{equation*}
f(\mathbf{x}, t)=f(\mathbf{f}(\mathbf{X}, t), t)=F(\mathbf{X}, t) \tag{7.1.5}
\end{equation*}
$$

but in fluid mechanics, the mapping $\mathbf{f}$ may not be known and this interpretation is not particularly useful.

In Chapter 4, Lagrangian finite elements were discussed. In Lagrangian finite element implementations, the finite element mesh convects with the material. The advantages of Lagrangian finite elements include the ease of tracking material interfaces and boundaries as well as the more straight-forward treatment of constitutive equations. Among the disadvantages of a Lagrangian formultation include the severe distortions that the elements may undergo as they deform with the material resulting in a deterioration of performance due to ill-conditioning. Nevertheless, Lagrangian finite elements prove extremely useful in large deformation problems in solid mechanics and are most widely used in solid mechanics. Eulerian finite elements are most often used in fluid mechanics for the same reasons that Eulerian representations of the equations of continuum mechanics are used,
i.e., there is often no well-defined reference configuration and the motion from a reference configuration is often not known explicitly. In Eulerian finite elements, the elements are fixed in space and material convects through the elements. Eulerian finite elements thus undergo no distortion due to material motion; however the treatment of constitutive equations and updates is complicated due to the convection of material through the elements. Eulerian elements may also lack resolution in the most highly deforming regions of the body.

The aim of ALE finite element formulations is to capture the advantages of both Lagrangian and Eulerian finite elements while minimizing the disadvantages. As the name suggests, ALE formulations are based on a description of the equations of continuum mechanics which is an arbitrary combination of the Lagrangian and Eulerian descriptions. The word arbitrary here means that the description (or specific combination of Lagrangian and Eulerian character) may be specified freely by the user. Of course, a judicious choice of the ALE motion is required if severe mesh distortions are to be eliminated. Suitable choices of the ALE motion will be discussed. Before introducing the ALE finite element formulation, it is useful to first consider some preliminary topics in continuum mechanics which were not covered in Chapter 3 and which provide the basis for the subsequent finite element implementation of the ALE methodology.

### 7.2 ALE Continuum Mechanics

### 7.2.1 Mesh Displacement, Mesh Velocity, and Mesh Acceleration

In figure (7.1), the motion $\mathbf{x}=\mathbf{f}(\mathbf{X}, t)$ is indicated as a mapping of the body from the reference configuration $\Omega_{0}$ to the current or spatial configuration $\Omega$. To introduce the ALE formulation, we now consider an alternative reference region $\hat{\Omega}$ as shown. We note that this region need not be an actual configuration of the body. Our objective is to show how the governing equations and kinematics for the body may be referred to this reference configuration and then how to use this description to formulate the ALE finite elements.

Points $\mathbf{C}$ in the reference region, $\hat{\Omega}$, are mapped to points $\mathbf{x}$ in the spatial region, $\Omega$ via the mapping

$$
\begin{equation*}
\mathbf{x}=\hat{\mathbf{f}}(\mathbf{c}, t) \tag{7.2.6}
\end{equation*}
$$

This mapping $\hat{\mathbf{f}}$ will ultimately play an important role in the ALE finite element formulation. At this point, it is regarded as an arbitrary mapping (although it will be assumed to be invertible) of the region $\hat{\Omega}$ to the region $\Omega$. The left hand side of (7.2.6) gives the mapping $\hat{\mathbf{f}}$ as a function of $\mathbf{C}$ and $t$. By virtue of (7.2.6), and (7.1.1), we have

$$
\begin{equation*}
\mathbf{x}=\hat{\mathbf{f}}(\mathbf{C}, t)=\mathbf{f}(\mathbf{X}, t) \tag{7.2.7}
\end{equation*}
$$

which states that $\mathbf{x}$ in the Eulerian representation, $\mathbf{c}$ in the ALE representation, and $\mathbf{X}$ in the Lagrangian representation are mapped into $\mathbf{x}$ (spatial coordinates) at time $t$. It is noted that even though the ALE mapping $\hat{\mathbf{f}}$ is different from the material mapping $\mathbf{f}$, the spatial coordinates $\mathbf{x}$ are the same.

In particular, if $\mathbf{c}$ is chosen to be the Lagrangian coordinate $\mathbf{X}, \hat{\mathbf{f}}$ becomes the material mapping $\mathbf{f}$ so that Eq.(7.2.7) becomes Eq.(7.1.1). A natural question arises: what is the ALE mapping $\hat{\mathbf{f}}$ if $\mathbf{c}$ is chosen to be the spatial coordinate $\mathbf{x}$ ? In this situation it is intuitive to think that Eq.(7.2.6) becomes:

$$
\begin{equation*}
\mathbf{x}=\hat{\mathbf{f}}(\mathbf{x}, t) \tag{7.2.8}
\end{equation*}
$$

Therefore, $\hat{\mathbf{f}}$ is an identity mapping and it is not a function of time. As a result, we may define the material and mesh velocities in the spatial coordinate form:

$$
\begin{equation*}
\mathbf{v}(\mathbf{x}, t)=\left.\frac{\partial \mathbf{f}(\mathbf{X}, t)}{\partial t}\right|_{\mathbf{x}} \tag{7.2.9a}
\end{equation*}
$$

and

$$
\begin{equation*}
\overline{\mathbf{v}}(\mathbf{x}, t)=\left.\frac{\partial \hat{\mathbf{f}}(\mathbf{c}, t)}{\partial t}\right|_{c} \tag{7.2.9b}
\end{equation*}
$$

It is noted that the right hand sides of Eqs.(7.2.9) are simply the definitions of material and mesh velocities, whereas the complete knowledge of the functions of the material and mesh velocites are often the solutions to the ALE continuum conservation equations. It is also understood that the mesh velocity, $\overline{\mathbf{v}}(\mathbf{x}, t)$, is equal to zero for an Eulerian description. We now assume that the two velocity equations are given so that with the definitions of the material motion, Eq.(7.1.1), and the mesh motion, Eq.(7.2.6), a set of first order boundary value equations are obtained:

$$
\begin{equation*}
\left.\frac{\partial \mathbf{f}(\mathbf{X}, t)}{\partial t}\right|_{\mathbf{X}}=\mathbf{v}(\mathbf{f}(\mathbf{X}, t), t) \tag{7.2.10a}
\end{equation*}
$$

and

$$
\begin{equation*}
\left.\frac{\partial \hat{\mathbf{f}}(\mathbf{c}, t)}{\partial t}\right|_{\chi}=\overline{\mathbf{v}}(\hat{\mathbf{f}}(\mathbf{c}, t), t) \tag{7.2.10b}
\end{equation*}
$$

The objective of Eqs.(7.2.10) is: given the material velocity function $\mathbf{v}(\mathbf{x}, t)$, and the mesh velocity function $\overline{\mathbf{v}}(\mathbf{x}, t)$, find the material mapping $\mathbf{f}(\mathbf{X}, t)$ and the ALE mapping $\hat{\mathbf{f}}(\mathbf{C}, t)$ such that Eqs.(7.2.10) are satisfied with the following initial conditions :

$$
\begin{equation*}
\mathbf{f}(\mathbf{X}, 0)=\mathbf{X}_{0} \tag{7.2.11a}
\end{equation*}
$$

and

$$
\begin{equation*}
\hat{\mathbf{f}}(\mathbf{c}, 0)=\mathbf{X}_{0} \tag{7.2.11b}
\end{equation*}
$$

With the stated initial boundary value problem, the above raised questions regarding the ALE mapping $\hat{\mathbf{f}}$ when $\mathbf{C}$ is chosen to be $\mathbf{x}$ can be answered by choosing $\mathbf{C}=\mathbf{x}$ ( an Eulerian description, implying $\overline{\mathbf{v}}(\mathbf{x}, t)=\mathbf{0})$, so that

$$
\begin{equation*}
\overline{\mathbf{v}}(\hat{\mathbf{f}}(\mathbf{C}, t), t)=\left.\frac{\partial \hat{\mathbf{f}}(\mathbf{x}, t)}{\partial t}\right|_{\mathbf{x}}=\left.\frac{\partial \mathbf{I}(\mathbf{x})}{\partial t}\right|_{\mathbf{x}}=\mathbf{0} \tag{7.2.12}
\end{equation*}
$$

Hence, Eq.(7.2.10b) becomes

$$
\begin{equation*}
\left.\frac{\partial \hat{\mathbf{f}}(\mathbf{c}, t)}{\partial t}\right|_{\mathbf{c}}=\left.\frac{\partial \hat{\mathbf{f}}(\mathbf{x}, t)}{\partial t}\right|_{\mathbf{x}}=\mathbf{0} \tag{7.2.13}
\end{equation*}
$$

and therefore,

$$
\hat{\mathbf{f}}(\chi, t)=\text { constant },
$$

is determined from the initial conditions. By choosing $\mathbf{x}=\mathbf{X}_{0}$, Eq.(7.2.8) becomes the identity mapping so that

$$
\begin{equation*}
\mathbf{x}=\hat{\mathbf{f}}(\mathbf{x}, t)=\mathbf{I}(\mathbf{x}) \tag{7.2.14}
\end{equation*}
$$

Thus $\hat{\mathbf{f}}$ is indeed an identity mapping when $\mathbf{c}=\mathbf{x}$.
In the finite element implementation of the ALE formulation, a mesh is defined with respect to the configuration $\hat{\Omega}$. The motion $\hat{\mathbf{f}}(\mathbf{C}, t)$ is used to describe the motion of the mesh and, as mentioned earlier, is chosen so as to reduce the effects of mesh distortion. For this reason, we also refer to $\hat{\mathbf{f}}(\mathbf{c}, t)$ as the mesh motion. In this sense, we introduce the mesh displacement, $\hat{\mathbf{u}}$, for points in $\hat{\Omega}$ through

$$
\begin{equation*}
\mathbf{x}=\hat{\mathbf{f}}(\mathbf{c}, t)=\mathbf{c}+\hat{\mathbf{u}}(\mathbf{c}, t) \tag{7.2.8}
\end{equation*}
$$

Consistent with this terminology, we also introduce the mesh velocity and acceleration fields for points in $\hat{\Omega}$ as follows

$$
\begin{equation*}
\hat{\mathbf{v}}=\left.\frac{\partial \hat{\mathbf{f}}(\mathbf{c}, t)}{\partial t}\right|_{/ \mathbf{c} /}=\left.\frac{\partial \hat{\mathbf{u}}(\mathbf{c}, t)}{\partial t}\right|_{[\mathbf{c}]} \tag{7.2.9}
\end{equation*}
$$

and

$$
\begin{equation*}
\hat{\mathbf{a}}=\left.\frac{\partial \hat{\mathbf{v}}(\mathbf{c}, t)}{\partial t}\right|_{[\mathbf{c} /}=\left.\frac{\partial^{2} \hat{\mathbf{u}}(\mathbf{c}, t)}{\partial t^{2}}\right|_{[\mathbf{c} /]} \tag{7.2.10}
\end{equation*}
$$

This expression for velocity could be written as

$$
\begin{equation*}
\hat{\mathbf{v}}=\left.\frac{\partial \mathbf{x}}{\partial t}\right|_{[\mathbf{c} /} \tag{7.2.11}
\end{equation*}
$$

However, in the interest of clarity in the ALE formulation, we refrain from this notation as it eliminates the distinction between the mapping or motion (in this case $\hat{\mathbf{f}}$ ) and the value
of the mapping, $\mathbf{x}$. For purely Eulerian or Lagrangian descriptions, however, this distinction in notation can be dropped with little loss of clarity.

Referring to (7.2.7), it can be seen that points in the reference configuration can be identified as

$$
\begin{equation*}
\mathbf{c}=\mathbf{y}(\mathbf{X}, t)=\hat{\mathbf{f}}^{-1} \circ \mathbf{f}(\mathbf{X}, t) \tag{7.2.12}
\end{equation*}
$$

although we will have little occasion to use this relation. Instead, we will make use of the previously defined mappings and the chain rule where necessary. A schematic diagram representing these descriptions is shown in Fig. 7.1 and a summary of the kinematics for a general ALE formulation with Lagrangian and Eulerian formulations shown as special cases is given in Table 7.1.


Fig 7.1 Mappings between Lagrangian, Eulerian, ALE descriptions

| Description |  | General ALE | Lagrangian | Eulerian |
| :---: | :---: | :---: | :---: | :---: |
| Motion | Material | $\mathbf{x}=\mathbf{f}(\mathbf{X}, t)$ | $\mathbf{x}=\mathbf{f}(\mathbf{X}, t)$ | $\mathbf{x}=\mathbf{f}(\mathbf{X}, t)$ |
|  | Mesh | $\mathbf{x}=\mathbf{f}(\mathbf{c}, t)$ | $\begin{gathered} \mathbf{x}=\mathbf{f}(\mathbf{X}, t) \\ (\mathbf{c}=\mathbf{X}, \hat{\mathbf{f}}=\mathbf{f}) \end{gathered}$ | $\begin{gathered} \mathbf{x}=I(\mathbf{x}) \\ (\mathbf{c}=\mathbf{x}, \hat{\mathbf{f}}=\mathbf{I}) \end{gathered}$ |
| Displacement | Material | $\mathbf{u}=\mathbf{x}-\mathbf{X}$ | $\mathbf{u}=\mathbf{x}-\mathbf{X}$ | $\mathbf{u}=\mathbf{x}-\mathbf{X}$ |
|  | Mesh | $\hat{\mathbf{u}}=\mathbf{x}-\mathbf{c}$ | $\hat{\mathbf{u}}=\mathbf{x}-\mathbf{X}=\mathbf{u}$ | $\hat{\mathbf{u}}=\mathbf{x}-\mathbf{x}=\mathbf{0}$ |
| Velocity | Material | $\mathbf{v}=\mathbf{u}_{, t[X]}$ | $\mathbf{v}=\mathbf{u}_{, t[X]}$ | $\mathbf{v}=\mathbf{u}_{, t[X]}$ |
|  | Mesh | $\hat{\mathbf{v}}=\hat{\mathbf{u}}_{, t[\chi]}$ | $\hat{\mathbf{v}}=\hat{\mathbf{u}}_{, t[\mathbf{X}]}=\mathbf{v}$ | $\hat{\mathbf{v}}=\hat{\mathbf{u}}_{, t[\mathbf{x}]}=\mathbf{0}$ |
| Acceleration | Material | $\mathbf{a}=\mathbf{v}_{, t[X]}$ | $\mathbf{a}=\mathbf{v}_{, t[X]}$ | $\mathbf{a}=\mathbf{v}_{, t[X]}$ |


| Mesh | $\hat{\mathbf{a}}=\hat{\mathbf{v}}_{, t[\chi]}$ | $\hat{\mathbf{a}}=\hat{\mathbf{v}}_{, t[X]}=\mathbf{a}$ | $\hat{\mathbf{a}}=\hat{\mathbf{v}}_{, t[\mathbf{x}]}=\mathbf{0}$ |
| :---: | :---: | :---: | :---: |

Table 7.1 Kinematics for a general ALE formulation with Lagrangian and Eulerian formulations shown as special cases.

### 7.2.2 Time Derivatives

In the balance laws, the material time derivative of a function appears. For a given scalar function, $f=f(\mathbf{x}, t)=F(\mathbf{X}, t)$, the material time derivative, and the spatial derivative or spatial gradient of $f$ which appears in continuum conservation laws are defined as:

$$
\begin{equation*}
\dot{f}=\frac{D}{D t} F(\mathbf{X}, t)=\left.\frac{\partial F(\mathbf{X}, t)}{\partial t}\right|_{[X]}=f(\mathbf{x}(\mathbf{X}, t), t)_{, t[X]} \tag{7.2.14a}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{\partial f}{\partial x_{i}} \equiv f_{, i} \quad \text { or } \quad \operatorname{grad}_{x} f \equiv \operatorname{grad} f \tag{7.2.14b}
\end{equation*}
$$

respectively. The subscript $x$ denotes partial differentiation with respect to $x$. These two important shorthand notations will be used subsequently.

### 7.2.3 Convective Velocity

Although functions $f(\mathbf{x}, t)$ are usually given in terms of $\mathbf{x}$ and $t$, it is convenient in ALE mechanics to express the function $f$ in terms of $\mathbf{c}$ in the finite element formulation since the initial input coordinates $\mathbf{C}$ are fixed in the finite element mesh.

In general, by composition of mapping, $f$ can be expressed as a function of $\mathbf{X}$ and $t$, denoted by $F$; a function of $\mathbf{x}$ and $t$, denoted by $f$; or a function of $\mathbf{C}$ and $t$, denoted by $\hat{f}$. That is

$$
\begin{equation*}
f=F(\mathbf{X}, t)=f(\mathbf{x}, t)=\hat{f}(\mathbf{C}, t) \tag{7.2.15}
\end{equation*}
$$

These are different functions, which represent the same field. The material time derivative can be expressed for the different descriptions as follow:

$$
\begin{align*}
\frac{D f}{D t}=\dot{F}= & \text { material time derivative }=F_{, t[\mathbf{X}]}(\mathbf{X}, t)  \tag{7.2.16a}\\
& =f_{, t[x]}+\left.\frac{\partial f}{\partial x_{i}} \frac{\partial x_{i}}{\partial t}\right|_{[X]}=f_{, t[x]}+f_{, i} v_{i} \tag{7.2.16b}
\end{align*}
$$

$$
\begin{equation*}
=\hat{f}_{, t[\chi]}+\left.\frac{\partial \hat{f}}{\partial \chi_{i}} \frac{\partial \chi_{i}}{\partial t}\right|_{[X]}=\hat{f}_{, t[\chi]}+\frac{\partial \hat{f}}{\partial \chi_{i}} w_{i} \quad(\mathbf{c}, t) \tag{7.2.16c}
\end{equation*}
$$

where $w_{i}$ is the particle velocity in the referential coordinates and may be defined explicitely as

$$
\begin{equation*}
w_{i}=\left.\frac{\partial \chi_{i}}{\partial t}\right|_{[X]} \tag{7.2.16d}
\end{equation*}
$$

. In Eq. $(7.2 .16 \mathrm{c})$, the variable $\mathbf{C}$ is not defined explicitly in terms of $\mathbf{X}$ and $t$ through the components of $\mathbf{x}$, but is given in terms of the material motion $\mathbf{f}$ and also of the mesh motion $\hat{\mathbf{f}}$. That is:

$$
\begin{equation*}
x_{j}=\phi_{j}(\mathbf{X}, t)=\hat{\phi}_{j}(\mathbf{c}, t) \tag{7.2.17}
\end{equation*}
$$

Differentiating with respect to time while holding $\boldsymbol{X}$ fixed gives:

$$
\begin{equation*}
x_{j, t[\mathbf{X}]}=v_{j}=\left.\frac{\partial \hat{\phi}_{j}}{\partial t}\right|_{[\chi]}+\left.\frac{\partial \hat{\phi}_{j}}{\partial \chi_{i}} \frac{\partial \chi_{i}}{\partial t}\right|_{[\mathbf{X}]}=\hat{v}_{j}+\left.\frac{\partial x_{j}}{\partial \chi_{i}} \frac{\partial \chi_{i}}{\partial t}\right|_{[\mathbf{X}]} \tag{7.2.18}
\end{equation*}
$$

The second term on the right hand side can be rearranged to yield:

$$
\begin{equation*}
\left.\frac{\partial x_{j}}{\partial \chi_{i}} \frac{\partial \chi_{i}}{\partial t}\right|_{[\mathbf{X}]}=\frac{\partial x_{j}}{\partial \chi_{i}} w_{i}=v_{j}-\hat{v}_{j} \equiv c_{j} \tag{7.2.19}
\end{equation*}
$$

where $c_{j}$ are the components of the convective velocity c. Applying the chain rule to Eq. (7.2.16c) and employing Eq. (7.2.19) yields:

$$
\begin{equation*}
\frac{D f}{D t}=\dot{F}=\hat{f}_{, t[\chi]}+\left.\frac{\partial f}{\partial x_{j}} \frac{\partial x_{j}}{\partial \chi_{i}} \frac{\partial \chi_{i}}{\partial t}\right|_{[X]}=\hat{f}_{, t[\chi]}+f_{, j} c_{j} \tag{7.2.20a}
\end{equation*}
$$

or in vector notation:

$$
\begin{equation*}
\frac{D f}{D t}=\hat{f}_{, t[\chi]}+\mathbf{c} \cdot \operatorname{grad} f=\hat{f}_{, t[\chi]}+\mathbf{c} \cdot \nabla_{x} f \tag{7.2.20b}
\end{equation*}
$$

It can be shown that Eq. (7.2.20a) reduces to Eqs. (7.2.16a) and (7.2.16b) when $\mathbf{c}=\mathbf{X}(\mathbf{c}=\mathbf{0})$ and $\mathbf{c}=\mathbf{x}(\mathbf{c}=\mathbf{v})$, respectively. The former is known as the Lagrangian description, whereas the latter is the Eulerian description. Equation (7.2.20) is the material time derivative of $f$ in a referential (i.e., ALE) description.

## Example

The comparison of the Lagrangian, Eulerian, and ALE descriptions is pictorially depicted in Fig. 7.2 by a 4 -node one dimensional finite element mesh. The finite element nodes and the material points are denoted by circles $(O)$ and solid $\operatorname{dots}(O)$, respectively. The normalized coordinates are: $X_{1}=0, X_{2}=1, X_{3}=2$, and $X_{4}=3$; and normalied time is between 0 and 1. In Chapter 3, the Lagrangian and Eulerian descriptions were described as shown in Figs. 7.2(a) and (c). To illustrate the ALE description, as shown in Fig. 7.2(b), the motion of the material points is described by:

$$
\begin{equation*}
x=\phi(X, t)=\left(1-X^{2}\right) t+X t^{2}+X \tag{7.2.13a}
\end{equation*}
$$

In order to regulate the mesh motion, the four mesh nodes are spaced uniformly based on the end points of the material motion, that is, $\phi\left(X_{1}, t\right)$ and $\phi\left(X_{4}, t\right)$. Therefore, the mesh motion can be described by a linear Lagrange polynomial:

$$
\begin{equation*}
x=\hat{\phi}(\chi, t)=\frac{\chi-\chi_{1}}{\chi_{4}-\chi_{1}} \phi\left(X_{1}, t\right)+\frac{\chi-\chi_{4}}{\chi_{1}-\chi_{4}} \phi\left(X_{4}, t\right) \tag{7.2.13b}
\end{equation*}
$$

Combining Eqs. (7.2.13a) and (b) yields :

$$
x=\frac{\chi-\chi_{1}}{\chi_{4}-\chi_{1}}\left[\left(1-X_{1}^{2}\right) t+X_{1}\left(t^{2}+1\right)\right]+\frac{\chi-\chi_{4}}{\chi_{1}-\chi_{4}}\left[\left(1-X_{4}^{2}\right) t+X_{4}\left(t^{2}+1\right)\right]
$$

Therefore, we have:
material displacement:

$$
u=x-X=\left(1-X^{2}\right) t+X t^{2}+X-X=\left(1+X-X^{2}\right) t
$$

material velocity:

$$
v=\left.\frac{\partial u}{\partial t}\right|_{X}=\left(1+X-X^{2}\right)
$$

material acceleration:

$$
a=\left.\frac{\partial v}{\partial t}\right|_{X}=0
$$

mesh displacement:

$$
\hat{u}=x-\chi=\frac{\chi-\chi_{1}}{\chi_{4}-\chi_{1}}\left[\left(1-X_{1}^{2}\right) t+X_{1}\left(t^{2}+1\right)\right]+\frac{\chi-\chi_{4}}{\chi_{1}-\chi_{4}}\left[\left(1-X_{4}^{2}\right) t+X_{4}\left(t^{2}+1\right)\right]-\chi
$$

mesh velocity:

$$
\hat{v}=\left.\frac{\partial \hat{u}}{\partial t}\right|_{\chi}=\frac{\chi-\chi_{1}}{\chi_{4}-\chi_{1}}\left[\left(1-X_{1}^{2}\right)+2 X_{1} t\right]+\frac{\chi-\chi_{4}}{\chi_{1}-\chi_{4}}\left[\left(1-X_{4}^{2}\right)+2 X_{4} t\right]
$$

mesh acceleration:

$$
\hat{a}=\left.\frac{\partial \hat{v}}{\partial t}\right|_{\chi}=\frac{2 X_{1}\left(\chi-\chi_{1}\right)}{\chi_{4}-\chi_{1}}+\frac{2 X_{4}\left(\chi-\chi_{4}\right)}{\chi_{1}-\chi_{4}}
$$

The ALE mapping from the material domain to the reference domain is given by:

$$
\chi=\psi(X, t)=\frac{\left[\left(1-X^{2}\right) t+X\left(t^{2}+1\right)\right]\left(\chi_{4}-\chi_{1}\right)+\chi_{1} \phi\left(X_{1}, t\right)-\chi_{4} \phi\left(X_{4}, t\right)}{\phi\left(X_{1}, t\right)-\phi\left(X_{4}, t\right)}
$$

The particle velocity and acceleration in the referential coordinates may then be computed using Eq. (7.2.16d) and its time deravitive, respectively.

Comparing the two motions above, even though both motions give the same range of $x$, the two mappings are quite different as shown in Eqs. (7.2.13a,b) and Fig.7.2b.


Fig 7.2 Comparison of Lagrangian, Eulerian, and ALE descriptions

### 7.4 Updated ALE Balance Laws in Referential Description

To derive the updated ALE balance laws analogous to those of the Lagrangian description, it is convenient to first use the Lagrangian equations given in Chapter 3 and then apply Eq. (7.2.20) to the material time derivatives to obtain the ALE conservation laws. Consequently, the only difference between the updated Lagrangian and updated ALE formulations is in the material time derivative terms. For completeness, the total ALE formulations are given in Appendix 7.1.

### 7.4.1 Conservation of Mass (Equation of Continuity) in ALE

The continuity equation is given by:

$$
\begin{equation*}
\dot{\rho}+\rho v_{j, j}=0 \tag{7.4.1}
\end{equation*}
$$

Applying the material time derivative operator Eqs. (7.2.20) to Eq. (7.4.1), the continuity equation becomes:

$$
\begin{equation*}
\rho_{, t[\chi]}+\rho_{, j} c_{j}+\rho v_{j, j}=0 \tag{7.4.2a}
\end{equation*}
$$

or in vector form:

$$
\begin{equation*}
\rho_{, t[\chi]}+\mathbf{c} \cdot \operatorname{grad} \rho+\rho \nabla \cdot \mathbf{v}=0 \tag{7.4.2b}
\end{equation*}
$$

where $\nabla \cdot \mathbf{v}$ is the divergence of $\mathbf{v}$ in index free notation.
An alternate way of deriving the continuity equation is to employ the Reynolds transport theorem (given in Chapter 3) and using the divergence theorem to give:

$$
\begin{equation*}
\int_{\Omega}\left[\left.\frac{\partial \rho}{\partial t}\right|_{x}+\frac{\partial\left(\rho v_{i}\right)}{\partial x_{i}}\right] d \Omega=0 \tag{7.7.14b}
\end{equation*}
$$

Assuming there are no discontinuities in the linear momentum, an application of the chain rule yields

$$
\begin{equation*}
\int_{\Omega}\left\lceil\left.\frac{\partial \rho}{\partial t}\right|_{x}+v_{i} \frac{\partial \rho}{\partial x_{i}}+\rho \frac{\partial v_{i}}{\partial x_{i}}\right] d \Omega=0 \tag{7.7.14c}
\end{equation*}
$$

Observing that the first two terms yield the material time derivative of $\rho$ and hence using Eq. (7.2.20), Eq. (7.7.14c) becomes:

$$
\begin{equation*}
\int_{\Omega}\left\lceil\left.\frac{\partial \rho}{\partial t}\right|_{\chi}+c_{i} \frac{\partial \rho}{\partial x_{i}}+\rho \frac{\partial v_{i}}{\partial x_{i}}\right\rfloor d \Omega=0 \tag{7.7.14d}
\end{equation*}
$$

and since $\Omega$ is arbitrarily chosen, it follows that:

$$
\begin{equation*}
\left.\frac{\partial \rho}{\partial t}\right|_{\chi}+c_{i} \frac{\partial \rho}{\partial x_{i}}+\rho \frac{\partial v_{i}}{\partial x_{i}}=0 \quad \text { in } \Omega \tag{7.7.15}
\end{equation*}
$$

which is identical to Eq.(7.4.2). It is noted that if there is a discontinuity, we cannot apply the chain rule to the linear momentum since there is a jump in $\rho v_{i}$ hence we have to employ the conservative form, Eq.(7.7.14b) instead of the non-conservative form, Eq.(7.4.1)

### 7.4.2 Conservation of Linear Momentum in ALE

The conservative form of the momentum equation is given as:

$$
\begin{equation*}
\frac{D}{D t}\left(\rho v_{i}\right)+\left(\rho v_{i}\right) v_{j, j}=\sigma_{j i, j}+\rho b_{i} \tag{7.4.6a}
\end{equation*}
$$

It was shown in Chapter 3 that if there are no discontinuities, then the non-conservative form of the momentum equation can be obtained by applying the chain rule to $\rho v_{i}$. With the help of the continuity equation, $\operatorname{Eq}(7.4 .1)$, we obtain

$$
\rho \dot{v}_{i}=\sigma_{j i, j}+\rho b_{i}
$$

Similarly, after applying the material time derivative operator Eqs.(7.2.20) to Eq.(7.4.6a), the momentum equation becomes:

$$
\begin{equation*}
\rho\left\{v_{i, t[\chi]}+c_{j} v_{i, j}\right\}=\sigma_{j i, j}+\rho b_{i} \tag{7.4.8a}
\end{equation*}
$$

or, in index free notation:

$$
\begin{equation*}
\rho\left\{\mathbf{v}_{, t|x|}+\mathbf{c} \cdot \operatorname{grad} \mathbf{v}\right\}=\operatorname{div}(\sigma)+\rho \mathbf{b} \tag{7.4.8b}
\end{equation*}
$$

It is a simple exercise that by applying the material time derivative operator to the energy equation derived in Chapter 3, and show that the non-conservative form of the energy equation is:

$$
\rho \dot{E}=\left(v_{i} \sigma_{i j}\right)_{, j}+b_{i} v_{i}+\left(k_{i j} \theta_{, j}\right)_{, i}+\rho s
$$

### 7.5 Formal Statement of the Updated ALE Governing Equations in NonConservative Form (Strong Form) in Referential Description

In the equations given below, (7.6.2), $k_{i j}$ and $v_{i}$ are the components of the thermal conductivity matrix and convective heat transfer coefficients, respectively; $\theta_{0}$ is the ambient temperature; $b_{i}$ are the components of the body force; and $s$ is the heat source. The objective of the initial/boundary-value problem is to find the following functions:

$$
\begin{array}{ll}
\mathbf{u}(\mathbf{X}, t) & \text { material displacement } \\
\sigma(\mathbf{x}, t) & \text { Cauchy stress tensor } \\
\theta(\mathbf{x}, t) & \text { thermodynamic temperature } \\
\hat{\mathbf{u}}(\mathbf{C}, t) & \text { mesh displacement } \tag{7.6.1d}
\end{array}
$$

and

$$
\rho(\mathbf{x}, t) \quad \text { density }
$$

such that they satisfy the following field and state equations shown in Box 7.1:

## Strong Form of Updated ALE Governing Equations in Referential

 Description
## Continuity Equation

$$
\begin{equation*}
\dot{\rho}+\rho v_{k, k}=0 \quad \text { or } \quad \rho_{, t[\chi]}+\rho_{, i} c_{i}+\rho v_{k, k}=0 \tag{7.6.2a}
\end{equation*}
$$

Momentum Equations

$$
\begin{equation*}
\rho \dot{v}_{i}=\sigma_{j i, j}+\rho b_{i} \quad \text { or } \quad \rho\left(v_{i, t[\chi]}+v_{i, j} c_{j}\right)=\sigma_{j i, j}+\rho b_{i} \tag{7.6.2b}
\end{equation*}
$$

## Energy Equation

$$
\rho \dot{E}=\left(v_{i} \sigma_{i j}\right)_{, j}+b_{i} v_{i}+\left(k_{i j} \theta_{, j}\right)_{, i}+\rho s \quad \text { or }
$$

$$
\begin{equation*}
\rho\left(E_{, t[\chi]}+E_{, i} c_{i}\right)=\left(v_{i} \sigma_{i j}\right)_{, j}+b_{i} v_{i}+\left(k_{i j} \theta_{, j}\right)_{, i}+\rho s \tag{7.6.2c}
\end{equation*}
$$

Equations of State
supplemented by the constitutive equations given in Chapter 6.

## Natural Boundary Conditions

$$
\begin{array}{ll}
t_{i}(\mathbf{x}, t)=n_{j}(\mathbf{x}, t) \sigma_{j i}(\mathbf{x}, t) & \text { on } \partial \Gamma_{x}^{t} \\
q_{i}(\mathbf{x}, t)=-k_{i j}(\theta, \mathbf{x}, t) \theta_{, j}(\mathbf{x}, t)+v_{i}(\theta, t)\left(\theta-\theta_{0}\right) & \text { on } \partial \Gamma_{x}^{t \theta} \tag{7.6.2h}
\end{array}
$$

## Essential Boundary Conditions

$$
\begin{array}{ll}
u_{i}(\mathbf{x}, t)=\bar{u}_{i}(\mathbf{x}, t) & \text { on } \partial \Gamma_{x}^{g} \\
\theta(\mathbf{x}, t)=\bar{\theta}(\mathbf{x}, t) & \text { on } \partial \Gamma_{x}^{g \theta} \tag{7.6.2j}
\end{array}
$$

## Initial Conditions

$$
\begin{array}{ll}
\mathbf{u}(\mathbf{X}, 0)=\mathbf{u}_{0}, & \hat{\mathbf{u}}(\mathbf{c}, 0)=\hat{\mathbf{u}}_{0} \\
\mathbf{v}(\mathbf{X}, 0)=\mathbf{v}_{0}, & \hat{\mathbf{v}}(\mathbf{c}, 0)=\hat{\mathbf{v}}_{0} \tag{7.6.21}
\end{array}
$$

Mesh Motion
$\hat{\mathbf{u}}(\mathbf{C}, t)=$ a given representation except, perhaps, on part of the boundary.

Box 7.1 Strong Form of Updated ALE Governing Equations in Referential Description
Prior to developing the weak form and Petrov-Galerkin finite element discretization of the ALE continuity and momentum equations outlined above, it is most instructive to digress briefly and formally acquaint the reader with the general Petrov-Galerkin method. In doing so, it hoped that the necessity and power of such an approach will become apparent and that an intuitive feeling for the physics involved will be brought to light. The following section is designed to fulfill this requirement after which our development of the ALE equations will resume.

### 7.14 Introduction to the Petrov-Galerkin Method

In this section, streamline upwinding by a Petrov Galerkin method(SUPG) is formulated. Prior to the development of this method, the need for an upwinding scheme was motivated by an examination of the classical advection-diffusion equation. The advection-diffusion equation is a useful model for studying the momentum since it corresponds to a linearization of the transport equation. A closed form solution for the discrete steady-state advection diffusion equation will be obtained. It will be shown that the solution is oscillatory when a parameter of the mesh, known as the Peclet number, exceeds a critical value. Next, a Petrov Galerkin method will be developed which eliminates these oscillations.

Consider the linear advection-diffusion equation:

$$
\begin{equation*}
\frac{\partial \phi}{\partial t}+\mathbf{u} \cdot \nabla \phi-v \nabla^{2} \phi=0 \tag{7.14.1}
\end{equation*}
$$

where $\phi$ is the dependent variable, $v$ is the kinematic viscosity, and $\mathbf{u}$ is a given velocity field. For the steady state case, $\frac{\partial \phi}{\partial t}=0$. So, the steady state equation is:

$$
\begin{equation*}
\mathbf{u} \cdot \nabla \phi-v \nabla^{2} \phi=0 \tag{7.14.2}
\end{equation*}
$$

For the study of special numerical instabilities, we restrict Eq. (7.14.2) to one dimension so that:

$$
\begin{equation*}
u \frac{d \phi}{d x}=v \frac{d^{2} \phi}{d x^{2}} \tag{7.14.3}
\end{equation*}
$$

Equation (7.14.3) with boundary conditions:

$$
\begin{equation*}
\phi(0)=0 \text { and } \phi(L)=1 \tag{7.14.4}
\end{equation*}
$$

is a two-point boundary value problem on the domain $0 \leq x \leq L$.
It is easy to verify that the exact solution to Eqs. (7.14.3) and (7.14.4) is:

$$
\begin{equation*}
\phi(x)=\frac{1-e^{u x / v}}{1-e^{u L / v}} \tag{7.14.5}
\end{equation*}
$$

7.14.1 The Galerkin Finite Element Approximation of the Advection-Diffusion Equation Letting the test function be $w(x)$, multiplying Eq. (7.14.3) by $w$, and integrating over the domain gives

$$
\begin{equation*}
\int_{\Omega} w\left(u \frac{d \phi}{d x}-v \frac{d^{2} \phi}{d x^{2}}\right) d x=0 \tag{7.14.6}
\end{equation*}
$$

Integrating by parts and making use of the divergence theorem, the weak form of the one dimensional advection-diffusion equation, Eq. (7.14.3), is

$$
\begin{equation*}
\int_{\Omega} w u \frac{d \phi}{d x} d x+\int_{\Omega} v w_{, x} \phi_{, x} d x=0 \tag{7.14.7}
\end{equation*}
$$

with $w \in \mathcal{U}_{0}$. The domain $(0, L)$ is then divided into equally sized linear finite elements, $\Omega^{e}$, on which the finite element approximation is given by:

$$
\begin{equation*}
\left(\int_{\Omega^{e}} u N_{a} N_{b, x} d x\right) \phi_{b}+\left(\int_{\Omega^{e}} v N_{a, x} N_{b, x} d x\right) \phi_{b}=0 \quad a, b=1,2 \tag{7.14.8}
\end{equation*}
$$

where $N_{a}$ and $N_{b}$ are the linear finite elements shape functions. This can be written in matrix component form as

$$
\begin{equation*}
N_{a b} \phi_{b}+K_{a b} \phi_{b}=0 \quad a, b=1,2 \tag{7.14.9a}
\end{equation*}
$$

where the convective matrix is given as:

$$
\begin{equation*}
N_{a b}=\int_{x_{e}}^{x_{e+1}} u N_{a} N_{b, x} d x \tag{7.14.9b}
\end{equation*}
$$

and the diffusion matrix is:

$$
\begin{equation*}
K_{a b}=\int_{x_{e}}^{x_{c+1}} v N_{a, x} N_{b, x} d x \tag{7.14.9c}
\end{equation*}
$$

It is a simple exercise to show that, when using linear finite element shape functions,

$$
\mathbf{N}=\frac{u}{2}\left[\begin{array}{ll}
-1 & 1  \tag{7.14.10}\\
-1 & 1
\end{array}\right] \quad \mathbf{K}=\frac{v}{\Delta x}\left[\begin{array}{cc}
1 & -1 \\
-1 & 1
\end{array}\right]
$$

After assembly, the equation for the $j$ th node is :

$$
\begin{equation*}
u\left(\frac{\phi_{j+1}-\phi_{j-1}}{2 \Delta x}\right)-v\left(\frac{\phi_{j+1}-2 \phi_{j}+\phi_{j-1}}{\Delta x^{2}}\right)=0 \tag{7.14.11a}
\end{equation*}
$$

which is exactly the central difference approximation. It is convenient to normalize the above equation, so that:

$$
\begin{equation*}
\frac{u \Delta x}{2 v}\left(\phi_{j+1}-\phi_{j-1}\right)-\left(\phi_{j+1}-2 \phi_{j}+\phi_{j-1}\right)=0 \tag{7.14.11b}
\end{equation*}
$$

The Peclet number, $P_{e}$, may then be defined as:

$$
\begin{equation*}
P_{e}=\frac{u \Delta x}{2 v} \tag{7.14.12}
\end{equation*}
$$

In terms of the Peclet number, Eq. (7.14.11b) then becomes:

$$
\begin{align*}
& P_{e}\left(\phi_{j+1}-\phi_{j-1}\right)-\left(\phi_{j+1}-2 \phi_{j}+\phi_{j-1}\right)=0  \tag{7.14.13a}\\
& \left(P_{e}-1\right) \phi_{j+1}+2 \phi_{j}-\left(P_{e}+1\right) \phi_{j-1}=0 \tag{7.14.13b}
\end{align*}
$$

Ignoring the boundary conditions, Eq. (7.14.13) can be put into the standard matrix notation by expanding the $j$ th term in Eq. (7.14.13):


The solution to the discrete finite difference Eq. (7.14.13) can be obtained by assuming:

$$
\begin{equation*}
\phi\left(x_{j}\right) \equiv \phi_{j}=e^{a x_{j}}=e^{a(j \Delta x)}=e^{(a \Delta x) j} \equiv z^{j} \tag{7.14.15}
\end{equation*}
$$

Where $z=e^{a \Delta x}$ and $a$ is an unknown coefficient to be determined. By the definition in Eq. (7.14.15), the $j+1$ th and $j$ - 1 th terms of $\phi$ are:

$$
\begin{align*}
& \phi_{j+1}=e^{a(j+1) \Delta x}=e^{a j \Delta x} e^{a \Delta x}=z^{j+1}  \tag{7.14.16a}\\
& \phi_{j-1}=e^{a(j-1) \Delta x}=e^{a j \Delta x} e^{-a \Delta x}=z^{j-1} \tag{7.14.16b}
\end{align*}
$$

Substituting Eqs. (7.14.16) into Eq. (7.14.13) yields:

$$
\begin{equation*}
\left(P_{e}-1\right) z^{j+1}+2 z^{j}-\left(P_{e}+1\right) z^{j-1}=0 \tag{7.14.17}
\end{equation*}
$$

Assuming that $z^{j-1} \neq 0$ and dividing the above equation by $z^{j-1}$, Eq. (7.14.17) becomes:

$$
\begin{equation*}
\left(P_{e}-1\right) z^{2}+2 z-\left(P_{e}+1\right)=0 \tag{7.14.18}
\end{equation*}
$$

The roots for Eq. (7.14.18) are:

$$
\begin{equation*}
z=1 \text { or } z=\frac{1+P_{e}}{1-P_{e}} \tag{7.14.19}
\end{equation*}
$$

Recalling that $\phi_{j}=z^{j}$, the solution to Eq. (7.14.13) takes the form:

$$
\begin{equation*}
\phi_{j}=c_{1}+c_{2}\left(\frac{1+P_{e}}{1-P_{e}}\right)^{j} \tag{7.14.20}
\end{equation*}
$$

where $c_{1}$ and $c_{2}$ are coefficients to be determined from the boundary conditions. Since the exact solution to Eq. (7.14.3) is given by Eq. (7.14.5), the exact solution of $\phi$, evaluated at $x=x_{j}$, has the form of:

$$
\begin{equation*}
\phi\left(x_{j}\right)=\frac{1}{1-e^{u L / v}}\left[1-e^{u x_{j} / v}\right]=c_{1}+c_{2} e^{\frac{u}{v} j \Delta x} \tag{7.14.21}
\end{equation*}
$$

Comparing the finite difference solution Eq. (7.14.20) with the exact solution, Eq. (7.14.21), it can be concluded that:
(i) If the Peclet number is less than one, i.e., $\left|P_{e}\right|<1$, then the discrete solution will have a solution similar to the exponential solution as given in the exact solution since $\left(\frac{1+P_{e}}{1-P_{e}}\right)^{j}>0$.
(ii) If the Peclet number is greater than one, i.e., $P_{e}>1$, then the discrete solution becomes:

$$
\left(\frac{1+P_{e}}{1-P_{e}}\right)^{j}=(-m)^{j} \quad \text { with } m>0
$$

Hence nodal oscillations occur because $\phi_{j}$ is positive or negative depending on whether $j$ is even or odd, respectively. To illustrate these nodal oscillations, we consider the one
dimensional advection-diffusion equation as given in Eq.(7.14.3) with boundary conditions (7.14.4). The plots below compare the exact solution with finite element solutions for the cases of both no upwinding and full upwinding. In all cases, 80 elements were used with an element Peclet number of 300 .
7.14.2 Ramification of Nodal Oscillation by the Petrov-Galerkin Formulation

Recall the weak form of Eq. (7.14.3):

$$
\begin{equation*}
\int_{\Omega} w\left(u \frac{d \phi}{d x}-v \frac{d^{2} \phi}{d x^{2}}\right) d x=0 \tag{7.14.27}
\end{equation*}
$$

The Petrov-Galerkin formulation for Eq. (7.14.3) is obtained by replacing the test function $w$ by $\tilde{w}$, where $\tilde{w}$ is defined as:

$$
\begin{equation*}
w \rightarrow \tilde{w} \equiv \underbrace{w}_{\text {Galerkintestfunction }}+\underbrace{\alpha \frac{\Delta x}{2} \frac{d w}{d x}(\operatorname{sign} u)} \tag{7.14.28}
\end{equation*}
$$

Replacing $w$ by $\tilde{w}$, Eq. (7.14.27) becomes:

$$
\begin{equation*}
\int_{\Omega} \tilde{w}\left(u \frac{d \phi}{d x}-v \frac{d^{2} \phi}{d x^{2}}\right) d x=0 \tag{7.14.29}
\end{equation*}
$$

Note that $w \in \mathcal{U}_{0}$ and $\tilde{w} \notin \mathcal{U}_{0}$. The parameter $\alpha$ is to be determined so as to eliminate oscillations for $P_{e}>1$ and hopefully get accurate solutions; in one dimension, it is possible to select $\alpha$ so as to obtain exact values of the solution at the nodes. Substituting the definition of $\tilde{w}$, Eq. (7.14.28), into Eq. (7.14.29) yields:

$$
\begin{align*}
0=\int_{\Omega} \tilde{w}\left(u \frac{d \phi}{d x}-v \frac{d^{2} \phi}{d x^{2}}\right) d x & =\underbrace{\int_{\Omega} w\left(u \frac{d \phi}{d x}-v \frac{d^{2} \phi}{d x^{2}}\right) d x}_{\text {GalerkinTerm }} \\
& +\underbrace{\sum_{e=1}^{N_{e}} \int_{\Omega^{e}} \alpha \frac{\Delta x}{2} \frac{d w}{d x}(\operatorname{sign} u)\left(u \frac{d \phi}{d x}-v \frac{d^{2} \phi}{d x^{2}}\right) d x}_{\text {Upwind Petrov-Galerkin Term }}
\end{align*}
$$

After integrating by parts (and using $w(0)=w(\mathrm{~L})=0$ be construction), Eq. (7.14.30) becomes:

$$
0=\int_{\Omega} w u \frac{d \phi}{d x} d x+\int_{\Omega} v \frac{d w}{d x} \frac{d \phi}{d x} d x
$$

$$
\begin{equation*}
+\sum_{e=1}^{N_{e}} \int_{\Omega^{e}} u \frac{\alpha \Delta x}{2}(\operatorname{sign} u) \frac{d w}{d x} \frac{d \phi}{d x} d x-\sum_{e=1}^{N_{e}} \int_{\Omega^{e}} v \frac{\alpha \Delta x}{2}(\operatorname{sign} u) \frac{d w}{d x} \frac{d^{2} \phi}{d x^{2}} d x \tag{7.14.31}
\end{equation*}
$$

The above equation is known as upwinding Petrov-Galerkin formulation(Brooks \& Hughes, 1978). It is noted that in this formulation the second derivative of $\phi$ is required. Further, the free parameter $\alpha$ is determined in the following section after the presentation of an alternative formulation which only requires the first derivative of $\phi$.

### 7.14.3 An Alternative Derivation of the Upwind/Petrov-Galerkin Formulation

This section outlines an alternative derivation of the upwind formulation. Motivated by a desire to necessitate low order continuity restrictions on the trial functions, we begin by starting with the one dimensional advection-diffusion equation as before. The equation is then multiplied by a test function $\tilde{w}$ and integrated over the domain $\Omega$ (following the traditional weak form development) thereby yielding

$$
\int_{\Omega} \tilde{w}\left(u \frac{d \phi}{d x}-v \frac{d^{2} \phi}{d x^{2}}\right) d x=0
$$

Examining the second term on the left hand side in more detail, our goal is to remove one derivative from the trial function and place it on the test function, $\tilde{w}$, thereby relaxing trial function continuity requirements. To this end, we integrate by parts in the familiar manner as follows:

$$
I \equiv \phi \int_{\Omega} \tilde{w} v \frac{d^{2} \phi}{d x^{2}} d x=\int_{\Omega}\left(\tilde{w} v \frac{d \phi}{d x}\right)_{, x} d x-\int_{\Omega} \frac{d \tilde{w}}{d x} v \frac{d \phi}{d x} d x
$$

Applying the divergence theorem and the substituting in the definition of $\tilde{w}$ gives:

$$
\begin{aligned}
& I=\left.\tilde{w} v \frac{d \phi}{d x}\right|_{o} ^{L}-\int_{\Omega} \frac{d \tilde{w}}{d x} v \frac{d \phi}{d x} d x=\left.\left[w+\alpha \frac{\Delta x}{2} \frac{d w}{d x}(\operatorname{sign} u)\right\rfloor v \frac{d \phi}{d x}\right|_{0} ^{L}-\int_{\Omega} \frac{d \tilde{w}}{d x} v \frac{d \phi}{d x} d x \\
& \left.=\left.\alpha \frac{\Delta x}{2} \frac{d w}{d x}(\operatorname{sign} u) \frac{d \phi}{d x}\right|_{0} ^{L}-\int_{\Omega} \frac{d \tilde{w}}{d x} v \frac{d \phi}{d x} d x \quad \text { (since } \quad w(o)=0 \quad \text { and } w(L)=0\right)
\end{aligned}
$$

Combining the above results with the advection term yields the following alternative weak form:

$$
\int_{\Omega}\left[\tilde{w} u \frac{d \phi}{d x}+\frac{d \tilde{w}}{d x} v \frac{d \phi}{d x}\right] d x-\alpha \frac{\Delta x}{2} \frac{d w}{d x}\left(\left.\operatorname{sign} u \vee \frac{d \phi}{d x}\right|_{0} ^{L}=0\right.
$$

Now it is apparent that removal of a trial function derivative gives rise to a boundary integral term which was not present in the Petrov-Galerkin formulation presented earlier. In the particular case when $\tilde{w}$ is defined as in Eq. (7.14.28), it is straightforward to show that this alternative formulation yields the same results as the formulation presented in the previous section. To see this, substitute the explicit expression for $\tilde{w}$ into the equation giving

$$
\begin{aligned}
& \int_{\Omega}\left[w+\alpha \frac{\Delta x}{2} \frac{d w}{d x}(\operatorname{sign} u)\right\rfloor u \frac{d \phi}{d x} d x+\int_{\Omega}\left[\frac{d w}{d x}+\alpha \frac{\Delta x}{2} \frac{d^{2} w}{d x^{2}}(\operatorname{sign} u)\right] v \frac{d \phi}{d x} d x \\
& -\alpha \frac{\Delta x}{2} \frac{d w}{d x}\left(\left.\operatorname{sign} u \wedge \frac{d \phi}{d x}\right|_{0} ^{L}=0\right.
\end{aligned}
$$

Using integrating by parts on the fourth term gives

$$
\int_{\Omega}\left[\alpha \frac{\Delta x}{2} \frac{d^{2} w}{d x^{2}}(\operatorname{sign} u)\right] v \frac{d \phi}{d x} d x=\left.\alpha \frac{\Delta x}{2}(\operatorname{sign} u) v \frac{d w}{d x} \frac{d \phi}{d x}\right|_{0} ^{L}-\int_{\Omega} \alpha \frac{\Delta x}{2}(\operatorname{sign} u) v \frac{d w}{d x} \frac{d^{2} \phi}{d x^{2}} d x
$$

which, upon rearrangement of the terms, yields the expression resulting from the previously presented Petrov-Galerkin formulation:

$$
\begin{aligned}
& \int_{\Omega} w u \frac{d \phi}{d x} d x+\int_{\Omega} v \frac{d w}{d x} \frac{d \phi}{d x} d x+\sum_{e=1}^{N_{e}} \int_{\Omega^{e}} \alpha \frac{\Delta x}{2}(\operatorname{sign} u) u \frac{d w}{d x} \frac{d \phi}{d x} d x \\
& -\sum_{e=1}^{N_{e}} \int_{\Omega^{e}} \alpha \frac{\Delta x}{2}(\operatorname{sign} u) v \frac{d w}{d x} \frac{d^{2} \phi}{d x^{2}} d x+\left.\alpha \frac{\Delta x}{2}(\operatorname{sign} u) v \frac{d w}{d x} \frac{d \phi}{d x}\right|_{0} ^{L}- \\
& \left.\alpha \frac{\Delta x}{2}(\operatorname{sign} u) v \frac{d w}{d x} \frac{d \phi}{d x}\right|_{0} ^{L}=0
\end{aligned}
$$

Following the cancelation of the last two terms, this equation is identical to that of the previous formulation, Eq.(7.14.31). As a result, we may select either formulation depending on which is more convenient computationally for the problem at hand.
Finally, it can be noted that when linear elements are used, $\frac{d^{2} \phi}{d x^{2}}=0$, so all terms involving second derivatitves vanish. Consequently, Eq. (7.14.31) may be written as:

$$
\begin{equation*}
0=\sum_{e} \int_{\Omega^{e}}\left\lceil w u \frac{d \phi}{d x}+v^{*} \frac{d w}{d x} \frac{d \phi}{d x}\right\rfloor d x \tag{7.14.32}
\end{equation*}
$$

where, $v^{*}$, which may be thought of as a sum of two viscosities will be defined below.

### 7.14.4 Parameter Determination and Further Analysis

To begin, we wish to shed additional light on the physical interpretation of Eq. (7.14.32). As a result, we make the following definitions:

$$
\begin{equation*}
v^{*}=v+\bar{v}=\text { total viscosity } \tag{7.14.33a}
\end{equation*}
$$

and

$$
\begin{equation*}
\bar{v}=\alpha u \frac{\Delta x}{2} \operatorname{sign}(u), \alpha \geq 0 \tag{7.14.33b}
\end{equation*}
$$

It then becomes clear that $\bar{v}$ may be thought of as an artificial viscosity which must be added to the "normal" flow viscosity, $v$, to ensure stability. That is, without this superficial damping which does not correspond to the physics of the problem, our numerical solution oscillates wildly thereby leading to physically meaningless results.

To define these viscosities in terms of various Peclet numbers, consider the following relationships:

$$
\begin{align*}
& v \Leftarrow P_{e}=\frac{u \Delta x}{2 v}  \tag{7.14.34a}\\
& \bar{v} \Leftarrow \bar{P}_{e}=\frac{u \Delta x}{2 \bar{v}}  \tag{7.14.34b}\\
& v^{*} \Leftarrow P_{e}^{*}=\frac{u \Delta x}{2 v^{*}} \tag{7.14.34c}
\end{align*}
$$

The relationships between Eq. (7.14.34a-c) can be expressed as:

$$
\begin{equation*}
\frac{1}{P_{e}^{*}}=\frac{2 v^{*}}{u \Delta x}=\frac{2 v}{u \Delta x}+\frac{2 \bar{v}}{u \Delta x}=\frac{1}{P_{e}}+\frac{1}{\overline{P_{e}}} \tag{7.14.35}
\end{equation*}
$$

In Eq. (7.14.35), if $P_{e}^{*}<1$, then

$$
\begin{equation*}
\frac{2 v}{u \Delta x}+\frac{2 \bar{v}}{u \Delta x}>1 \text { or } \frac{1}{\overline{P_{e}}}>1-\frac{1}{P_{e}} \tag{7.14.36}
\end{equation*}
$$

and the solution will not be oscillatory. From Eq. (7.14.13), the discrete equation in terms of $P_{e}{ }^{*}$ can be written as:

$$
\begin{equation*}
\left(P_{e}^{*}-1\right) \phi_{j+1}+2 \phi_{j}-\left(P_{e}^{*}+1\right) \phi_{j-1}=0 \tag{7.14.37}
\end{equation*}
$$

Recall that the discrete solution is oscillatory when $\phi_{N}=z^{N}$, the roots of Eq. (7.14.37) are :

$$
\begin{equation*}
z^{N}=c_{1}, c_{2}\left(\frac{1+P_{e}^{*}}{1-P_{e}^{*}}\right)^{N} \tag{7.14.38}
\end{equation*}
$$

From Eq. (7.14.21) and Eq. (7.14.38), we can solve $P_{e}^{*}$. That is,

$$
\begin{equation*}
\left(\frac{1+P_{e}^{*}}{1-P_{e}^{*}}\right)^{j}=e^{\left(\frac{u}{v} \Delta x\right) j} \tag{7.14.39}
\end{equation*}
$$

The RHS of Eq. (7.14.39) can be expressed in terms of $P_{e}$. That is:

$$
\begin{equation*}
\frac{1+P_{e}^{*}}{1-P_{e}^{*}}=e^{\frac{u}{v} \Delta x}=e^{2\left(\frac{u \Delta x}{2 v}\right)}=e^{2 P_{e}} \tag{7.14.40a}
\end{equation*}
$$

or

$$
\begin{equation*}
1-e^{2 P_{e}}=-P_{e}^{*}\left(1+e^{2 P_{e}}\right) \tag{7.14.40b}
\end{equation*}
$$

Therefore,

$$
\begin{equation*}
P_{e}^{*}=\frac{e^{2 P_{e}}-1}{e^{2 P_{e}}+1}=\frac{e^{P_{e}}-e^{-P_{e}}}{e^{P_{e}}+e^{-P_{e}}} \tag{7.14.41a}
\end{equation*}
$$

or

$$
\begin{equation*}
P_{e}^{*}=\tanh \left(P_{e}\right) \tag{7.14.41b}
\end{equation*}
$$

Substituting Eq. (7.14.41b) into Eq. (7.14.35) yields:

$$
\begin{equation*}
\frac{1}{\tanh \left(P_{e}\right)}=\frac{1}{P_{e}}+\frac{1}{\bar{P}_{e}} \tag{7.14.42a}
\end{equation*}
$$

or

$$
\begin{equation*}
\frac{1}{\overline{P_{e}}}=\operatorname{coth}\left(P_{e}\right)-\frac{1}{P_{e}} \tag{7.14.42b}
\end{equation*}
$$

Together, Eq. (7.14.34b) and Eq. (7.14.42b) may be combined to give:

$$
\begin{equation*}
\bar{P}_{e}=\frac{u \Delta x}{2 \bar{v}}=\left[\operatorname{coth}\left(P_{e}\right)-\frac{1}{P_{e}}\right]^{-1} \tag{7.14.43}
\end{equation*}
$$

Using Eq. (7.14.43), we can also express $\bar{v}$ in terms of $P_{e}$ :

$$
\begin{equation*}
\bar{v}=\frac{1}{2} u \Delta x\left[\operatorname{coth}\left(P_{e}\right)-\frac{1}{P_{e}}\right]=\alpha \frac{u \Delta x}{2} \operatorname{sign}(u) \tag{7.14.44a}
\end{equation*}
$$

so therefore,

$$
\begin{equation*}
\overline{\mathrm{v}}=\frac{1}{2} u \Delta x\left[\operatorname{coth}\left(P_{e}\right)-\frac{1}{P_{e}}\right]=\alpha \frac{u \Delta x}{2} \operatorname{sign}(u) \tag{7.14.44b}
\end{equation*}
$$

Finally, it becomes apparent that we may define the parameter $\alpha$ as:

$$
\begin{equation*}
\alpha=\operatorname{sign}(u)\left[\operatorname{coth}\left(P_{e}\right)-\frac{1}{P_{e}}\right] \tag{7.14.45}
\end{equation*}
$$

Note that when $\alpha=0$, Eq. (7.14.29) is simply like a central difference method and when $\alpha=1$, it is a full upwind Petrov-Galerkin formulation.

### 7.14.5 Streamline-Upwind/Petrov-Galerkin Formulation for Multiple Dimensions

The advection-diffusion equation in multiple dimension is:

$$
\begin{equation*}
\mathbf{u} \cdot \nabla \phi-\mathbf{v} \nabla^{2} \phi=0 \quad \text { in } \Omega \tag{7.14.45a}
\end{equation*}
$$

where the boundary conditions are:

$$
\begin{array}{ll}
\phi=g & \text { on } \partial \Omega_{g} \\
\mathbf{v} \nabla \phi \cdot \mathbf{n}=0 & \text { on } \partial \Omega_{i} \tag{7.14.45c}
\end{array}
$$

The weak form of the advection-diffusion equation for a streamline-upwind/PetrovGalerkin formulation of Eq. (7.14.45a) is obtained by multiplying Eq. (7.14.45a) by the test function $w$ and integrating over the domain $\Omega$

$$
\int w\left(\mathbf{u} \cdot \nabla \phi-\mathbf{v} \nabla^{2} \phi\right)=0
$$

Similar to the one-dimensional formulation presented above, let the Petrov-Galerkin test functions be defined by

$$
\begin{equation*}
w \rightarrow \tilde{w} \equiv \underbrace{w}_{\text {Galerkintesffunction }}+\underbrace{\tau \mathbf{u} \cdot \nabla w}_{\text {discontinuoustest function }} \tag{7.14.45d}
\end{equation*}
$$

thus giving

$$
\begin{equation*}
0=\int_{\Omega}(w+\tau \mathbf{u} \cdot \nabla w)\left(\mathbf{u} \cdot \nabla \phi-v \nabla^{2} \phi\right) d \Omega \tag{7.14.46a}
\end{equation*}
$$

where the stabilization parameter is now given by

$$
\begin{equation*}
\tau=|\alpha| \frac{h}{2 \mid u \|} \text { and } h \equiv \Delta x \tag{7.14.45b}
\end{equation*}
$$

where $\alpha$ is given by Eq.(7.14.33c). For a time dependent problem, the stabilization parameter can be set by:

$$
\begin{equation*}
\tau=|\alpha| \frac{\Delta t}{2} \tag{7.14.45c}
\end{equation*}
$$

Note that $w \in \mathcal{U} u_{0}$ and $\tilde{w} \notin \mathcal{U}_{0}$ as in the one dimensional case. Applying integration by parts and the divergence theorem, the weak form of Eq. (7.14.46a) can be shown to be:

$$
\begin{align*}
& 0=\underbrace{\int_{\Omega} w \mathbf{u} \cdot \nabla \phi d \Omega+\int_{\Omega} v \nabla w \cdot \nabla \phi d \Omega}_{\text {Galerkin terms }} \\
& +\underbrace{\sum_{e=1}^{N_{e}} \int_{\Omega^{e}} \tau(\mathbf{u} \cdot \nabla w)(\mathbf{u} \cdot \nabla \phi) d \Omega-\sum_{e=1}^{N_{e}} \int_{\Omega^{e}} \tau(\mathbf{u} \cdot \nabla w) v \nabla^{2} \phi d \Omega}_{\text {streamlineupwind stabilization terms }} \tag{7.14.46}
\end{align*}
$$

As can be seen from the above equation, the Petrov-Galerkin terms are simply the sum of the standard Galerkin terms plus the streamline upwind stabilization terms. Namely,

$$
\text { Petrov/ Galerkin = Galerkin }+ \text { Streamline Upwind Stabilization }
$$

The third term of Eq. (7.14.46) can be rewritten as:

$$
\begin{align*}
& \int_{\Omega^{e}} \tau(\mathbf{u} \cdot \nabla w)(\mathbf{u} \cdot \nabla \phi) d \Omega \\
& =\int_{\Omega^{e}} \tau\left[\left.\begin{array}{lll}
w_{, 1} & w_{, 2} & w_{, 3}
\end{array}\right|_{\left[\left.\begin{array}{l}
u_{1} \\
u_{2} \\
u_{3}
\end{array} \right\rvert\,\right.} \left\lvert\, \begin{array}{lll}
u_{1} & u_{2} & u_{3}\left[\begin{array}{l}
\phi_{, 1} \\
\phi_{, 2} \\
\phi_{, 3}
\end{array}\right] d \Omega
\end{array}\right.\right. \\
& \left.\left.\left.\left.\left.=\int_{\Omega^{e}} \tau\left[\begin{array}{lll}
w_{, 1} & w_{, 2} & w_{, 3}
\end{array}\right] \begin{array}{lll}
u_{1} u_{1} & u_{1} u_{2} & u_{1} u_{3}
\end{array}\right]\right\rceil_{\phi_{, 1}}\right\rceil{ }_{\left[\begin{array}{ll}
u_{2} u_{1} & u_{2} u_{2} \\
u_{3} u_{1} & u_{2} u_{3}
\end{array} u_{3} u_{2}\right.} u_{3} u_{3}\right]\left.\phi_{, 2}\right|_{d \Omega} \phi_{, 3}\right] \\
& =\int_{\Omega^{e}} \nabla w \cdot(\tilde{v} \nabla \phi) d \Omega \tag{7.14.47a}
\end{align*}
$$

where

$$
\left.\tilde{v}=\tau \left\lvert\, \begin{array}{lll}
\left\lceil u_{1} u_{1}\right. & u_{1} u_{2} & u_{1} u_{3}  \tag{7.14.47b}\\
u_{2} u_{1} & u_{2} u_{2} & u_{2} u_{3} \\
u_{3} u_{1} & u_{3} u_{2} & u_{3} u_{3}
\end{array}\right.\right]
$$

Substituting Eq. (7.14.47a) into Eq. (7.14.46) and ignoring the last second order term, Eq. (7.14.46) becomes:

$$
\begin{equation*}
0=\underbrace{\int_{\Omega} w \mathbf{u} \cdot \nabla \phi d \Omega+\sum_{e=1}^{N_{e}} \int_{\Omega^{e}} \nabla w(v \mathbf{1}}_{\text {Galerkin Term }}+\underset{\tilde{v}}{\text { artifical viscosity }} \underset{\tilde{v}}{ }) \nabla \phi d \Omega \tag{7.14.48}
\end{equation*}
$$

The artificial viscosity acts as a stabilization term which eliminates the oscillations resulting from a standard Galerkin formulation.
7.14.6 An Alternative Derivation of the Multiple Dimensional Streamline-Upwind/PetrovGalerkin Formulation
Paralleling section 7.14.3, this section outlines an alternative derivation of the multidimensional streamline-upwind/Petrov-Galerkine formulation presented above. To this end, we begin by starting with the multiple dimensional advection-diffusion equation as before. The equation is then multiplied by a test function $\tilde{w}$ and integrated over the domain $\Omega$ (following the traditional weak form development) thereby yielding

$$
\int \tilde{w}\left(\mathbf{u} \cdot \nabla \phi-\mathbf{v} \nabla^{2} \phi\right)=0
$$

Examining the second term on the left hand side in more detail, our goal is to remove one derivative from the trial function and place it on the test function, $\tilde{w}$, thereby relaxing trial function continuity requirements. To this end, we integrate by parts in the familiar manner as follows:

$$
I \equiv \int_{\Omega} \tilde{w} \mathbf{v} \nabla^{2} \phi d \Omega=\int_{\Omega} \nabla(\tilde{w} \mathbf{v} \nabla \phi) d \Omega-\int_{\Omega} \nabla \tilde{w} \mathbf{v} \nabla \phi d \Omega
$$

Applying the divergence theorem and the substituting in the definition of $\tilde{w}$ gives:

$$
\begin{aligned}
& I=\left.\tilde{w} \mathbf{v} \nabla \phi\right|_{\Gamma}-\int_{\Omega} \nabla \tilde{w} \mathbf{v} \nabla \phi d \Omega=\left.[w+\tau \mathbf{u} \cdot \nabla w] \mathbf{v} \nabla \phi\right|_{\Gamma}-\int_{\Omega} \nabla \tilde{w} \mathbf{v} \nabla \phi d \Omega \\
& =\left.\tau \mathbf{u} \cdot \nabla w \mathbf{v} \nabla \phi\right|_{\Gamma}-\int_{\Omega} \nabla \tilde{w} \mathbf{v} \nabla \phi d \Omega
\end{aligned}
$$

where the last line has been obtained by using the fact that $w \in \mathcal{U}_{0}$. Combining the above results with the advection term yields the following alternative weak form:

$$
\int_{\Omega}[\tilde{w} \mathbf{u} \cdot \nabla w+\nabla \tilde{w} \mathbf{v} \nabla \phi] d \Omega-\left.\tau \mathbf{u} \cdot \nabla w \mathbf{v} \nabla \phi\right|_{\Gamma}=0
$$

Now it is apparent that removal of a trial function derivative gives rise to a boundary integral term which was not present in the streamline-upwind/Petrov-Galerkin formulation
presented earlier. In the particular case when $\tilde{w}$ is defined as in Eq. (7.14.28), it is straightforward to show that this alternative formulation yields the same results as the formulation presented in the previous section.

Example Petrov-Galerkin formulation of the ALE momentum equation for a 1D 2-node linear displacement element may be done as follows.
The test function is chosen to be:
$\bar{N}_{I}=N_{I}+\alpha \frac{h}{2} \frac{d N_{I}}{d x} \operatorname{sign}(c)$
where $\alpha$ is the viscosity constant, $c$ is the convective velocity, $h$ is the element size, and $N_{I}$ is the shape function of the usual Galerkin form. If constant density is assumed, we obtain the Petrov-Galerkin form of the mass matrix in the ALE formulation, as:

$$
\left.\mathbf{M}=\rho h \left\lvert\, \begin{array}{ll}
\frac{1}{3}-\frac{\alpha}{2}\left(K_{2}-K_{1}\right) & \frac{1}{6}-\frac{\alpha}{2} K_{1} \\
\frac{1}{6}+\frac{\alpha}{2}\left(K_{2}-K_{1}\right) & \frac{1}{3}+\frac{\alpha}{2} K_{1}
\end{array}\right.\right]
$$

where

$$
\begin{aligned}
& K_{1}= \begin{cases}\frac{1}{2} \operatorname{sign}\left(c_{1}\right) & \text { if }\left(c_{1}=c_{2}\right) \\
\frac{1}{2}\left[\operatorname{sign}\left(c_{1}\right)-\operatorname{sign}\left(c_{2}\right)\right]\left(\frac{c_{1}}{c_{1}-c_{2}}\right)^{2}+\frac{1}{2} \operatorname{sign}\left(c_{2}\right) & \text { if }\left(c_{1} \neq c_{2}\right)\end{cases} \\
& K_{2}= \begin{cases}\operatorname{sign}\left(c_{1}\right) & \text { if }\left(c_{1}=c_{2}\right) \\
{\left[\operatorname{sign}\left(c_{1}\right)-\operatorname{sign}\left(c_{2}\right)\right]\left(\frac{c_{1}}{c_{1}-c_{2}}\right)+\operatorname{sign}\left(c_{2}\right)} & \text { if }\left(c_{1} \neq c_{2}\right)\end{cases}
\end{aligned}
$$

and $c_{1}$ and $c_{2}$ are the convective velocities at the 2 nodes.
Similar to that for convective-advective equation, we can rewrite the mass matrix as:
namely, the Petrov-Galerkin terms can be split into the sum of Galerkin terms and the streamline upwind stabilization terms.

### 7.3 Weak Form and Petrov-Galerkin Finite Element Discretization of the ALE Continuity and Momentum Equations:

As can be seen from Box 7.1, the only difference between the updated Lagrangian equations and the updated ALE equations is the interpretation of the material time derivative. Hence the weak form, and subsequently the Galerkin finite element
formulations, are identical to those derived in Chapter 4. However, note that the spatial domain now depends on how the mesh motion is updated, which is one of the key ingredients in the updated ALE formulation.

The variational equations corresponding to the conservation equations of Box 7.1 are obtained by multiplying by the test functions, $\delta \rho$ and $\delta v_{i}$, integrating over the spatial domain $\Omega$ and employing the divergence theorem to embed the traction force vector $t$ on the boundary $\Gamma^{t}$. Following the same procedures as in Chapter 4, we can achieve the following weak forms:

Continuity Equation:

$$
\begin{equation*}
\int_{\Omega} \delta \rho \dot{\rho} d \Omega=\int_{\Omega} \delta \rho \rho_{, t[x]} d \Omega+\int_{\Omega} \delta \rho c_{i} \rho_{, i} d \Omega=-\int_{\Omega} \delta \rho \rho v_{i, i} d \Omega \tag{7.16.2a}
\end{equation*}
$$

Momentum Equation:

$$
\begin{align*}
& \int_{\Omega} \delta v_{i} \rho \dot{v}_{i} d \Omega \\
& =\int_{\Omega} \delta v_{i} \rho v_{i, t[x]} d \Omega+\int_{\Omega} \delta v_{i} \rho c_{j} v_{i, j} d \Omega  \tag{7.16.2b}\\
& =-\int_{\Omega} \delta v_{i, j} \sigma_{i j} d \Omega+\int_{\Omega} \delta v_{i} \rho b_{i} d \Omega_{x}+\int_{\Gamma^{t}} \delta v_{i} \bar{t}_{i} d \Gamma
\end{align*}
$$

It is noted that because convective terms ( $\rho_{, i} c_{i}$ and $v_{i, j} c_{j}$ ) appeared in the continuity and momentum equations, a Galerkin finite element formulation will give rise to numerical difficulties. Therefore, in this section, the Petrov-Galerkin formulation will be employed to alleviate some of these difficulties. In a Petrov Galerkin finite element discretization, the current domain $\Omega$ is subdivided into elements. However, different sets of shape functions, $\boldsymbol{N}$ and $\boldsymbol{N}^{\rho}$ for the trial functions, and $\overline{\mathbf{N}}$ and $\overline{\mathbf{N}}^{\rho}$ for the test functions, will be used to interpolate the velocity and density, respectively. If $\overline{\mathbf{N}}=\mathbf{N}$, the Galerkin ALE formulations will be obtained. The choice of $\overline{\mathbf{N}}$ and $\overline{\mathbf{N}}^{\rho}$ to eliminate numerical oscillations will be described in section ????.

The finite element matrix equations corresponding to Eq.(7.16.2a,b) are :
Continuity equation:

$$
\begin{equation*}
\mathbf{M}^{\rho} \rho_{, t[\chi]}+\mathbf{L}^{\rho} \rho+\mathbf{K}^{\rho} \rho=0 \tag{7.16.3a}
\end{equation*}
$$

where $\mathbf{M}^{\rho}, \mathbf{L}^{\rho}, \mathbf{K}^{\rho}$ are generalized mass, convective, and stiffness matrices, respectively, for density under a reference description such that:

$$
\begin{align*}
& \mathbf{M}^{\rho}=\left[M_{I J}^{\mathrm{\rho}}\right]=\int_{\Omega} \bar{N}_{I}^{\rho} N_{J}^{\mathrm{\rho}} d \Omega  \tag{7.16.3b}\\
& \mathbf{L}^{\rho}=\left[L_{I J}^{\mathrm{\rho}}\right]=\int_{\Omega} \bar{N}_{I}^{\rho} c_{i} N_{J, i}^{\mathrm{\rho}} d \Omega \tag{7.16.3c}
\end{align*}
$$

$$
\begin{equation*}
\mathbf{K}^{\rho}=\left[K_{I J}^{\mathrm{\rho}}\right]=\int_{\Omega} \bar{N}_{I}^{\rho} v_{i, i} N_{J}^{\rho} d \Omega \tag{7.16.3d}
\end{equation*}
$$

Momentum Equation:

$$
\begin{equation*}
\mathbf{M} v_{, t[\chi]}+\mathbf{L v}+\mathbf{f}^{i n t}=\mathbf{f}^{e x t} \tag{7.16.4a}
\end{equation*}
$$

where $\mathbf{M}$ and $\mathbf{L}$ are generalized mass and convective matrices, respectively, for velocity under a reference description; while $\mathbf{f}^{i n t}$ and $\mathbf{f}^{e x t}$ are the internal and external force vectors respectively, such that:

$$
\begin{align*}
& \mathbf{M}=\left[M_{I J}\right]=\int_{\Omega} \rho \bar{N}_{I} N_{J} d \Omega  \tag{7.16.4b}\\
& \mathbf{L}=\left[L_{I J}\right]=\int_{\Omega} \rho \bar{N}_{I} c_{i} N_{J, i} d \Omega  \tag{7.16.4c}\\
& \mathbf{f}^{i n t}=\left[f_{i I}^{i n t}\right]=\int_{\Omega} \bar{N}_{I, j} \sigma_{i j} d \Omega  \tag{7.16.4d}\\
& \mathbf{f}^{e x t}=\left[f_{i I}^{e x t}\right]=\int_{\Omega} \rho \bar{N}_{I} b_{i} d \Omega+\int_{\Gamma^{t}} \bar{N}_{I} \bar{t}_{i} d \Gamma \tag{7.16.4e}
\end{align*}
$$

REMARK 7.16.1.4 The nonself-adjoint and nonlinear convective terms, $\mathbf{L}^{\rho}$ and $\mathbf{L}$, which appear in Eqs.(7.16.3c) and (7.16.4c) and characterize the ALE method, will inevitably pose difficulties.

REMARK 7.16.1.5 All the matrices and vectors defined in $\operatorname{Eqs}(7.16 .3)-(7.16 .6)$ are integrated over the spatial domain $\Omega$. Unlike the Lagrangian formulations, the spatial domain changes continuously throughout the computation. The mesh update procedure will be described in section ????.

In the subsequent development of this chapter, we shall divide the discussion of ALE formulations into four parts:

1) Updated ALE formulations for continuum material models without memory, that is, the evaluation of constitutive laws is independent of the strain history. A simple example of this kind of materials is a slightly compressible Newtonian fluid which will be discussed in section ???
2) Updated ALE formulations of continuum material models with memory, that is, the evaluation of constitutive laws is strain history dependent. This will be described in section ???.
3) The Petrov-Galerkin finite element method, of which the discretization of the transport term requires special treatment. For high velocities, if the mesh is not sufficiently refined, the Galerkin method gives rise to oscillatory solutions. To overcome this difficulty, various schemes, collectively known as upwinding, will be described in section ????.
4) The mesh update procedure, which is described in section ????.

The reason for the distinction between ALE formulations with and without memory is due to the difference between ALE material update procedures. Recall that the material time
derivative of a given function $f$ can be related to the referential time derivative by Eq.(7.2.20), that is

$$
\dot{f}=\frac{D f}{D t}=\hat{f}_{, t[\chi]}+c_{j} f_{, j}
$$

It is noted that $f$ can represent density $\rho$, velocity $\mathbf{v}$, energy $E$, Cauchy stress $\sigma$, etc., as appearing in the governing equations Box 7.1.

The next section will illustrate the difference between Lagrangian and ALE material update procedures.

### 7.3 ALE Material Updates

In the Lagrangian description, the updating of any material-related state variable is simple. Since the Lagrangian coordinate is always associated with the same material point, a Taylor series expansion in time may be used. Using first order accuracy gives:

$$
\begin{equation*}
F(\mathbf{X}, t+d t)=F(\mathbf{X}, t)+\left.d t \frac{\partial F}{\partial t}(\mathbf{X}, t)\right|_{\mathbf{x}}+\ldots \tag{7.3.1}
\end{equation*}
$$

However, in a referential description, updating of a material state variable introduces complicates. To illustrate this, we expand a state variable $\hat{f}(\chi, t)$ in a Taylor series:

$$
\begin{equation*}
\hat{f}(\chi, t+d t)=\hat{f}(\chi, t)+\left.d t \frac{\partial \hat{f}}{\partial t}(\chi, t)\right|_{\chi}+\ldots \tag{7.3.2}
\end{equation*}
$$

or, by referring everything to the particle $\mathbf{X}$ at time t (see Eq. (7.2.10)), as:

$$
\begin{equation*}
\hat{f}[\psi(\mathbf{X}, t+d t), t+d t]=\hat{f}[\psi(\mathbf{X}, t), t]+\left.d t \frac{\partial \hat{f}}{\partial t}[\psi(\mathbf{X}, t), t]\right|_{\chi=\psi(\mathbf{X}, t)}+\ldots \tag{7.3.3}
\end{equation*}
$$

which is equivalent to

$$
\begin{equation*}
F(\overline{\mathbf{X}}, t+d t)=F(\mathbf{X}, t)+\left.d t \frac{\partial F}{\partial t}(\mathbf{X}, t)\right|_{\mathbf{X}=\psi^{-1}(\chi, t)}+\ldots \tag{7.3.4a}
\end{equation*}
$$

where

$$
\begin{equation*}
\overline{\mathbf{X}}=\psi^{-1}(\chi, t+d t) \quad \text { and } \quad \mathbf{X}=\psi^{-1}(\chi, t) \tag{7.3.4b}
\end{equation*}
$$

Comparing Eq. (7.3.4a) with Eq. (7.3.1), even though the terms in the right hand side of the equations are the same, shows that $\mathbf{X}$ and $\overline{\mathbf{X}}$ are two different material particles, which
at $t$ and $t+d t$, respectively, have the same referential coordinates. Therefore, in a referential description, a simple updating technique, such as Eq. (7.3.2) cannot be used for material point-related variables, such as state variables in path-dependent materials; however, for homogeneous materials with no memory, such as the generalized Newtonian fluids, Eq. (7.3.2) can be implemented with no further complications. Further detail will be given on this subject later.

### 7.16 ALE Finite Element Method for Path-Dependent Materials

The purpose of this section is to provide a general formulation and an explicit computational procedure for nonlinear ALE finite element analyses. Emphasis is placed on the stress update procedure for path-dependent materials. First, after the general formulations for the ALE description are reviewed, according to strong form, weak form and finite element form, the most important part of the ALE application for path-dependent materials, the stress update procedure, is studied in detail. Formulations for regular Galerkin method, Streamline-upwind/Petrov-Galerkin(SUPG) method and operator split method are derived respectively. All the path-dependent state variables are updated with a similar procedure. Further, the stress update procedures are specified in 1-D case. And the matrices corresponding to these three methods are listed. Then, an explicit computational method and a flowchart are presented. Finally, elastic and elastic-plastic wave propagation examples are given.

### 7.16.1 Formulations for Updated ALE:

### 7.16.1.1 Strong Form Formulations for ALE:

In addition to continuity and momentum equations given in section ????, for the purpose of introducing path-dependent material model, the Cauchy stress may be decomposed into the deviatoric stress tensor $s_{i j}$ and the hydrostatic pressure $p$ such that

$$
\begin{equation*}
\sigma_{i j}=s_{i j}-p \delta_{i j} \tag{7.16.1c}
\end{equation*}
$$

and the components of the deviatoric stress term are given by the Jaumann rate constitutive equation

$$
\begin{equation*}
s_{i j, t[\chi]}+c_{k} s_{i j, k}=C_{i j k l} D_{k l}+s_{k j} W_{i k}+s_{k i} W_{j k} \tag{7.16.1d}
\end{equation*}
$$

Similarly the rate from of the equation of state is given by:

$$
\begin{equation*}
p_{, t[\chi]}+c_{i} p_{, i}=p(\rho) \tag{7.16.1e}
\end{equation*}
$$

In the above equations, $D_{i j}$ and $W_{i j}$ are of deformation tensor and the spin tensor, respectively; such that

$$
\begin{equation*}
D_{i j}=\frac{1}{2}\left(v_{i, j}+v_{j, i}\right) \tag{7.16.1f}
\end{equation*}
$$

$$
\begin{equation*}
W_{i j}=\frac{1}{2}\left(v_{i, j}-v_{j, i}\right) \tag{7.16.1g}
\end{equation*}
$$

and $C_{i j k l}$ is the material response tensor which relates any frame-invariant rate of the Cauchy stress to the velocity strain. Both geometric and material nonlinearities are included in the setting of Eqs. (7.16.1a-g).

REMARK 7.16.1.1 The right-hand sides of Eqs. (7.16.1a,b,d and e) remain the same for all descriptions.
REMARK 7.16.1.2. Eqs. (7.16.1a,b,d and e) are referred to as the "quasi-Eulerian" description (Belytschko and Kennedy, 1978) because these equations have a strong resemblance to the Eulerian equations. In particular, the Eulerian equations can be readily obtained by choosing $\mathbf{c}=\mathbf{v}$, i.e., $\chi=\mathbf{x}$.
REMARK 7.16.1.3. With the help of integration by part, Eq.(7.16.1d) is equivalent to the following equations:

$$
\begin{equation*}
s_{i j, t[\chi]}+y_{i j k, k}-c_{k, k} s_{i j}=C_{i j k l} D_{k l}+s_{k j} W_{i k}+s_{k i} W_{j k} \tag{7.16.1h}
\end{equation*}
$$

and

$$
\begin{equation*}
y_{i j k}=s_{i j} c_{k} \tag{7.16.1i}
\end{equation*}
$$

where $y_{i j k}$ is the stress-velocity product. In the following finite element computation, these two equations will replace Eq.(7.16.1d) in the weak form.

### 7.16.1.2 Weak Form of the ALE Equations:

Similar to the case for continuity and momentum equations, we may obtain the weak form of the constitutive equations:

$$
\begin{align*}
& \int_{\Omega} \delta s_{i j} s_{i j, t[\chi]} d \Omega+\int_{\Omega} \delta s_{i j} y_{i j k, k} d \Omega-\int_{\Omega} \delta s_{i j} c_{k, k} s_{i j} d \Omega \\
& =\int_{\Omega} \delta s_{i j} C_{i j k l} D_{k l} d \Omega+\int_{\Omega} \delta s_{i j}\left\{s_{k j} W_{i k}+s_{k i} W_{j k}\right\} d \Omega \tag{7.16.2c}
\end{align*}
$$

and

$$
\begin{equation*}
\int_{\Omega} \delta y_{i j k} y_{i j k} d \Omega=\int_{\Omega} \delta y_{i j k} s_{i j} c_{k} d \Omega \tag{7.16.2d}
\end{equation*}
$$

Equation of state

$$
\begin{equation*}
\int_{\Omega} \delta p p_{t,[\chi]} d \Omega+\int_{\Omega} \delta p c_{i} p_{, i} d \Omega=\int_{\Omega} \delta p p(\rho) d \Omega \tag{7.16.2e}
\end{equation*}
$$

Eq.(7.16.2a) represents the control volume form of material conservation. Eq.(7.16.2b) is a generalization of the principle of virtual work to the control volume form with the first integral brought in as d'Alembert forces.

### 7.16.1.3 Finite Element Discretization:

Similar to the finite element discretization for continuity and momentum equations, we can obtain the finite element form for constitutive equations by employing $N^{s}, N^{y}$ and $N^{p}$ as
sets of shape functions, and $\bar{N}^{s}, \bar{N}^{y}$ and $\bar{N}^{p}$ as corresponding sets of test functions to interpolate the deviatoric stress, stress-velocity product, and hydrostatic pressure respectively. Note that the test functions and the shape functions for deviatoric stresses are used only in the constitutive equations.

## Constitutive Equations:

$$
\begin{align*}
& \mathbf{M}^{s} \mathbf{s}_{, t[\chi]}+\mathbf{G}^{T} \mathbf{y}-\mathbf{D} \mathbf{s}=\mathbf{z}  \tag{7.16.5a}\\
& \mathbf{M}^{y} \mathbf{y}=\mathbf{L}^{y} \mathbf{s} \tag{7.16.5b}
\end{align*}
$$

where the superscript T denotes matrix transpose, $\mathbf{M}^{s}$ and $\mathbf{D}$ are the generalized mass and diffusion matrices for deviatoric stress respectively; $\mathbf{G}^{T} \mathbf{y}$ corresponds to the generalized convective term; $\mathbf{M}^{y}$ and $\mathbf{L}^{y}$ are the generalized mass and convective matrices for stressvelocity product respectively; and $\mathbf{z}$ is the generalized deviatoric stress vector such that

$$
\begin{align*}
& \mathbf{M}^{s}=\left[M_{I J}^{s}\right]=\int_{\Omega} \bar{N}_{I}^{s} N_{J}^{s} d \Omega  \tag{7.16.5c}\\
& \mathbf{G}^{T}=\left[G_{I J}^{T}\right]=\int_{\Omega} \bar{N}_{I}^{s} N_{J, x}^{y} d \Omega  \tag{7.16.5d}\\
& \mathbf{D}=\left[D_{I J}\right]=\int_{\Omega} \bar{N}_{I}^{s} c_{k, k} N_{J}^{s} d \Omega  \tag{7.16.5e}\\
& \mathbf{z}=\left[z_{i j I}\right]=\int_{\Omega} \bar{N}_{I}^{s} C_{i j k l} D_{k l} d \Omega+\int_{\Omega} \bar{N}_{I}^{s}\left\{s_{k j} W_{i k}+s_{k i} W_{j k}\right\} d \Omega  \tag{7.16.5f}\\
& \mathbf{M}^{y}=\left[M_{I J}^{y}\right]=\int_{\Omega} \bar{N}_{I}^{y} N_{J}^{y} d \Omega  \tag{7.16.5g}\\
& \mathbf{L}^{y}=\left[L_{I J}^{y}\right]=\int_{\Omega} \bar{N}_{I}^{s} c N_{J}^{s} d \Omega \tag{7.16.5h}
\end{align*}
$$

Equation of State:

$$
\begin{equation*}
\mathbf{M}^{p} \mathbf{p}_{, t[\chi]}+\mathbf{L}^{p} \mathbf{p}=\mathbf{u} \tag{7.16.6a}
\end{equation*}
$$

where $\mathbf{M}^{p}$ and $\mathbf{L}^{p}$ are the generalized mass and convective matrices for pressure respectively; and $\mathbf{u}$ is the generalized pressure vector, such that

$$
\begin{align*}
& \mathbf{M}^{p}=\left[M_{I J}^{p}\right]=\int_{\Omega} \bar{N}_{I}^{p} N_{J}^{p} d \Omega  \tag{7.16.6b}\\
& \mathbf{L}^{p}=\left[L_{I J}^{p}\right]=\int_{\Omega} \bar{N}_{I}^{p} c_{i} N_{J, i}^{p} d \Omega  \tag{7.16.6c}\\
& \mathbf{u}=\left[u_{I}\right]=\int_{\Omega} \bar{N}_{I}^{p} p(\rho) d \Omega \tag{7.16.6d}
\end{align*}
$$

REMARK 7.16.1.6 The stress-velocity product $\mathbf{y}$ is stored at each node as a vector with a dimension of (number of space dimensions) $\times$ (number of stress components). The diagonal form for $\mathbf{M}^{y}$ is considered by location the numerical integration points at nodes.

REMARK 7.16.1.7 The numerical integration of (7.16.3) and (7.16.4) has been discussed by Liu \& Belytschko(1983), Liu(1981) and Liu \& Ma(1982). A procedure for the stress update Eqs. (7.16.5a) and (7.16.5b) is presented in the next section to clarify the temporal integration for path-dependent materials. All the path-dependent quantities are updated analogous to $\operatorname{Eqs}(7.16 .5 \mathrm{a}$ ) and (7.16.5b). The Petrov-Galerkin formulation of the continuity and momentum equation derived in section 7.14 can be adopted here.

### 7.16.2 Stress Update Procedures:

### 7.16.2.1 Stress Update Procedure for Galerkin Method

The stress state in a path-dependent material depends on the stress history of the material point. A stress history can be readily treated in Lagrangian description because elements contain the same material points regardless of the deformation of the continuum; similarly, quadrature points at which stresses are computed in Lagrangian elements coincide with material points throughout the deformation. On the other hand, in an ALE description, a mesh point does not coincide with a material point so that the stress history needs to be convected by the relative velocity c, as indicated in Eq.(7.16.1d). Note that the spatial derivatives of the deviatoric stress are involved in the convection term.
When $\mathbf{C}^{-1}$ functions are used to interpolate the element stresses, the ambiguity of the stress derivatives at the element interface renders the calculation of the spatial derivatives of stress a difficult task. As mentioned in Remark 7.16.1.3, this is remedied by replacing Eq.(7.16.1d) by a set of coupled equations, Eqs.(7.16.1h) and (7.16.1i), and the corresponding matrix equations have been given in Eqs.(7.16.5a) and (7.16.5b). The stress-velocity product $\mathbf{y}$ can be eliminated by inverting $\mathbf{M}^{y}$ in $\mathrm{Eq}(7.16 .5 \mathrm{~b}$ ) and substituting into $\mathrm{Eq}(7.16 .5 \mathrm{a})$ :

$$
\begin{equation*}
\mathbf{M}^{s} \mathbf{s}_{, t[\chi]}+\mathbf{G}^{T}\left(\mathbf{M}^{y}\right)^{-1} \mathbf{L}^{y} \mathbf{s}-\mathbf{D s}=\mathbf{z} \tag{7.16.7a}
\end{equation*}
$$

where $\mathbf{G}^{T}\left(\mathbf{M}^{y}\right)^{-1} \mathbf{L}^{y} \mathbf{s}$ can be identified as the convective term, and the upwind techniques mentioned in Remark 7.16.1.4 should be applied to evaluate $\mathbf{L}^{y} \mathbf{s}$. When $\mathbf{c}=0$, i.e., $\chi=\mathbf{x}$, Eq.(7.16.7a) degenerates to the usual stress updating formula in the Lagrangian description,

$$
\begin{equation*}
\mathbf{M}^{s} \mathbf{s}_{, t[\chi]}=\mathbf{z} \tag{7.16.7b}
\end{equation*}
$$

REMARK 7.16.2.1. The conservation equations are listed here to show the similarity among the equations:

$$
\begin{equation*}
\mathbf{M}^{\rho} \rho_{, t[\chi]}+\mathbf{L}^{\rho} \rho+\mathbf{K}^{\rho} \rho=0 \tag{7.16.7c}
\end{equation*}
$$

$$
\begin{align*}
& \mathbf{M} \mathbf{v}_{, t[\chi]}+\mathbf{L v}+\mathbf{f}^{\text {int }}=\mathbf{f}^{e x t}  \tag{7.16.7d}\\
& \mathbf{M}^{s} \mathbf{s}_{, t[\chi]}+\mathbf{G}^{T}\left(\mathbf{M}^{y}\right)^{-1} \mathbf{L}^{y} \mathbf{s}-\mathbf{D s}=\mathbf{z} \tag{7.16.7e}
\end{align*}
$$

REMARK 7.16.2.2. In the above, Eq.(7.16.5b) is eliminated because it can be incorporated in Eq.(7.16.7e). This procedure is analogous to that by Liu \& Chang(1985) where a fluid-structure interaction algorithm is described.

REMARK 7.16.2.3. All the path-dependent material properties, such as yield strains, effective plastic strains, yield stresses, and back stresses, should be convected via Eq.(7.16.7e) with $s$ replaced by these properties, respectively, and with $\mathbf{z}$ appropriately modified.

### 7.16.2.2 Stress Update Procedure for SUPG Method:

In a nonlinear displacement finite element formulation, when applied to elastic-plastic materials with kinematic hardening, the velocities are stored at nodes while the stress histories, back stresses, and yield radii are available only at quadrature points. In order to establish the nodal values for the stress-velocity product, a weak formulation is a logical necessity. In addition, based on the one-dimensional study(Liu et. al. 1986), in which the upwind procedure is used to define this intermediate variable, the artificial viscosity technique(streamline upwind) is considered here as a generalization of this upwind procedure to multi-dimensional cases.
The relation for the stresses-velocity product of Eq.(7.16.1i), is modified to accommodate the artificial viscosity tensor $A_{i j k m}$ by

$$
\begin{equation*}
y_{i j k}=s_{i j} c_{k}-A_{i j k m, m} \tag{7.16.8a}
\end{equation*}
$$

The ingredients of the artificial viscosity tensor consist of a tensorial coefficient multiplied by the stress:

$$
\begin{equation*}
A_{i j k m}=\mu_{k m} s_{i j} \tag{7.16.8b}
\end{equation*}
$$

where the tensorial coefficient is constructed to act only in the flow direction (streamline upwind effect)

$$
\begin{equation*}
\mu_{k m}=\bar{\mu} c_{k} c_{m} / c_{n} c_{n} \tag{7.16.8c}
\end{equation*}
$$

and the scalar $\bar{\mu}$ is given by

$$
\begin{equation*}
\bar{\mu}=\sum_{i=1}^{N S D} \alpha_{i} c_{i} h_{i} / N S D \tag{7.16.8d}
\end{equation*}
$$

Here $h_{i}$ is the element length in the $i$-direction; NSD designates the number of space dimensions; and $\alpha_{i}$ is the artificial viscosity parameter given by

$$
\alpha_{i}=\left\{\begin{array}{lll}
\frac{1}{2} & \text { for } & c_{i}>0  \tag{7.16.8e}\\
-\frac{1}{2} & \text { for } & c_{i}<0
\end{array}\right.
$$

The weak form corresponding to Eq. (7.16.8a) can be obtained by multiplying by test function for the stress-velocity product and integrating over the spatial domain $\Omega$ :

$$
\begin{equation*}
\int_{\Omega} \delta y_{i j k} y_{i j k} d \Omega=\int_{\Omega} \delta y_{i j k} s_{i j} c_{k} d \Omega-\int_{\Omega} \delta y_{i j k} A_{i j k m, m} d \Omega \tag{7.16.9a}
\end{equation*}
$$

This equation may be written as

$$
\begin{equation*}
\int_{\Omega} \delta y_{i j k} y_{i j k} d \Omega=\int_{\Omega} \delta y_{i j k} s_{i j} c_{k} d \Omega+\int_{\Omega} \delta y_{i j k, m} A_{i j k m} d \Omega \tag{7.16.9b}
\end{equation*}
$$

by applying the divergence theorem and by assuming no traction associated with the artificial viscosity on the boundary. The expression for $A_{i j k m}$, Eq. ( 7.16 .8 b and c), can be substituted into this equation to yield

$$
\begin{equation*}
\int_{\Omega} \delta y_{i j k} y_{i j k} d \Omega=\int_{\Omega}\left(\delta y_{i j k}+\delta \bar{y}_{i j k}\right) s_{i j} c_{k} d \Omega \tag{7.16.9c}
\end{equation*}
$$

where

$$
\begin{equation*}
\delta \bar{y}_{i j k}=\delta y_{i j k, m} \bar{\mu} c_{m} / c_{n} c_{n} \tag{7.16.9d}
\end{equation*}
$$

can be viewed as a modification of the Galerkin finite element method because of the transport nature of the stress-velocity product.
The shape functions for the stress-velocity product can be chosen to be the standard $C^{0}$ functions. The number and position of quadrature points for Eq.(7.16.9c and d) should be selected to be the Gauss quadrature points, since the stress histories in Eq.(7.16.9c) are only available at these points.

Following the procedures given above, the stress-velocity product can be defined at each nodal point and it can be substituted into the constitutive equation of Eq.(7.16.5a) to calculate the rate of change of stresses with the same procedure as that of without artificial viscosity. Note that the interpolation functions for stresses are integrated over the spatial element domain. The task of selecting the number of quadrature points for the displacement finite element poses another important issue. For example, the locking phenomenon for fully integrated elements arises when the material becomes incompressible(Liu, et.al., 1985). While selective reduced integration can overcome this difficulty, it is just as costly as full quadrature. To alleviate this computational hurdle, the use of one-point quadrature combined with hourglass control is developed by Belytschko et.al. (1984). In addition, the nonlinear two-quadrature point element (Liu et.al., 1988) appears to be another candidate for large-scale computations because it exhibits nearly the same accuracy as the selective reduced integration element while with only one-third of the cost. These two kind of elements, as well as the other displacement elements, can be readily adopted in the ALE computations.
Following the procedure described by Liu et.al(1986), the displacement element is divided into M sub-domain, where M denotes the number of quadrature points. Each sub-domain is designated by $\Omega_{I}(I=1, \ldots, \mathrm{M})$, which contains the quadrature points $\mathbf{x}_{I}$, and no two sub-domains overlap. Associated with $\Omega_{I}$, a stress interpolation function $N_{I}^{s}$ is assigned and its value is prescribed to be unity only at the quadrature point $\mathbf{x}=\mathbf{x}_{I}$ such that

$$
\begin{equation*}
N_{I}^{s}\left(\mathbf{x}=\mathbf{x}_{I}\right)=1 \tag{7.16.10a}
\end{equation*}
$$

The test function in $\Omega_{I}$ is chosen to be the Dirac delta function

$$
\begin{equation*}
\bar{N}_{I}^{s}=\delta\left(\mathbf{x}-\mathbf{x}_{I}\right) \tag{7.16.10b}
\end{equation*}
$$

Substitution of these functions into the constitutive equation represents a mathematical requirement that the residual of the weal from vanishes at each collocative quadrature point. Because the collocation point is located right at the quadrature point, the algebraic equations resulting from Eq.(7.16.5a) can be easily worked out without numerical integration and given as below:
General mass matrix is

$$
\begin{equation*}
M_{I J}^{s}=\int_{\Omega} \bar{N}_{I}^{s} N_{J}^{s} d \Omega_{x}=\mathbf{I}_{M \times M} \tag{7.16.11a}
\end{equation*}
$$

where the subscripts $I$ and $J$ range from 1 to $M$, the number of stress quadrature points per element. The transpose of the divergence operator matrix reads,

$$
G_{I J}^{T}=\int_{\Omega} \bar{N}_{I}^{s} N_{J, x}^{y} d \Omega
$$

For 2D 4-node element, it will be:

$$
\left.\mathbf{G}^{T}=\left[\begin{array}{cccc}
N_{1, x}\left(\xi_{1}\right) & N_{1, y}\left(\xi_{1}\right) & N_{2, x}\left(\xi_{1}\right) & N_{2, y}\left(\xi_{1}\right) \\
N_{1, x}\left(\xi_{2}\right) & N_{1, y}\left(\xi_{2}\right) & N_{2, x}\left(\xi_{2}\right) & N_{2, y}\left(\xi_{2}\right)  \tag{7.16.11b}\\
\vdots & \vdots & \vdots & \vdots \\
N_{1, x}\left(\xi_{M}\right) & N_{1, y}\left(\xi_{M}\right) & N_{2, x}\left(\xi_{M}\right) & N_{2, y}\left(\xi_{M}\right)
\end{array}\right\} \begin{array}{cccc}
N_{3, x}\left(\xi_{1}\right) & N_{3, y}\left(\xi_{1}\right) & N_{4, x}\left(\xi_{1}\right) & N_{4, x}\left(\xi_{1}\right) \\
N_{3, x}\left(\xi_{2}\right) & N_{3, y}\left(\xi_{2}\right) & N_{4, x}\left(\xi_{2}\right) & N_{4, y}\left(\xi_{2}\right) \\
\vdots & \vdots & \vdots & \vdots
\end{array} \right\rvert\,
$$

The generalized diffusion matrix for stress is

$$
D_{I J}=\int_{\Omega} \bar{N}_{I}^{s} c_{k, k} N_{J}^{s} d \Omega
$$

or

$$
\left.\mathbf{D}=\left\lvert\, \begin{array}{llll}
\left\lceil c_{k, k}\left(\xi_{1}\right)\right. & & &  \tag{7.16.11c}\\
& c_{k, k}\left(\xi_{2}\right) & & \\
& & \ddots & \\
& & & c_{k, k}\left(\xi_{M}\right)
\end{array}\right.\right]_{M \times M}
$$

The generalized stress vector is

$$
z_{i j I}=\int_{\Omega} \bar{N}_{I}^{s}\left\{C_{i j k l} D_{k l}+\tau_{k j} W_{i k}+\tau_{k i} W_{j k}\right\} d \Omega
$$

or

$$
\left.\mathbf{z}=\left\lvert\, \begin{array}{c}
\left\lceil\left(C_{i j k l} D_{k l}+\tau_{k j} W_{i k}+\tau_{k i} W_{j k}\right)_{\xi_{1}}\right.  \tag{7.16.11d}\\
\mid\left(C_{i j k l} D_{k l}+\tau_{k j} W_{i k}+\tau_{k i} W_{j k}\right)_{\xi_{2}} \\
\vdots \\
\left(C_{i j k l} D_{k l}+\tau_{k j} W_{i k}+\tau_{k i} W_{j k}\right)_{\xi_{M}}
\end{array}\right.\right]_{M \times 1}
$$

### 7.16.2.3. Stress Update Procedure for Operator Split Method:

In addition to the methods shown in the last section to solve the fully coupled equations, another approach referred to as an operator split is an alternative way to apply FEM to solve this problem numerically(Benson, 1989). Conceptually, the approach is simple. "Splitting" stands for "decomposing" a set of PDE operators into several sets of simple PDE operators which will be solved sequentially. An operator split decouples the various physical phenomena in the governing equations to obtain simpler equations to be solved more easily. In exchange of certain loss of accuracy, the operator split offers a generic advantage: simpler equations leads to simpler and stable algorithms, specifically designed for each decoupled equation according to the different physical characteristics.

To illustrate the operator split concept, we consider the transport equation of one component of the Cauchy stress by:

$$
\begin{equation*}
\sigma_{t \mid \mathbf{x}]}=\sigma_{, t \mid x]}+c_{i} \boldsymbol{\sigma}_{i}=q \tag{7.16.12}
\end{equation*}
$$

where the expression of $q$ will be determined by the constitutive law. An example of $q$ is the right hand of Eq.(7.16.1h).
The operator split technique is to split Eq.(7.16.12) into 2 phases. The first phase of the operator split is to solve a Lagrangian step without considering the convective effect as below:

$$
\begin{equation*}
\sigma_{, t[\chi]}=q \tag{7.16.13}
\end{equation*}
$$

In this Lagrangian phase, it is integrated in time to update stresses from $\sigma^{(t)}$ (stress at time $t$ ) to $\sigma^{(L)}$ (denotation of the Lagrangian updated stress), neglecting the convective terms which is equivalent to assuming that mesh points $\chi$ move with material particles $\mathbf{X}$, that is $\chi=\mathbf{X}$. Thus this Lagrangian phase can proceed in the same way as the usual Updated Lagrangian procedure. In addition, it is well known that the stresses are obtained at Gauss points.
The second phase is to deal with the convective term that has not been taken into account during the Lagrangian phase, where the governing PDE is :

$$
\begin{equation*}
\sigma_{, t[\chi]}+c_{i} \sigma_{, i}=0 \tag{7.16.14}
\end{equation*}
$$

and during which phase, the stresses are updated from $\sigma^{(L)}$ to $\sigma^{(t+\Delta t)}$.
According to the two phases strategy above, the constitutive equation is split into the parabolic equation of the Lagrangian phase and the hyperbolic equation of the convection phase.
Here, we follow [12] to apply explicit methods to integrate the convection equation. To compute gradients of the stress fields on the surface of elements, two different approaches
have been taken: i) use an explicit smoothing procedure (Lax-Wendroff update); or (ii) use an algorithm that circumvents the computation of the stress gradient (Godunov-like technique).
(1) Lax-Wendroff update:

The key point of the Lax-Wendroff method is to replace the time derivatives of depending variables with spatial derivatives using the governing equations. For the partial differential equations of Eq.(7.16.14), we have:

$$
\begin{equation*}
\sigma_{, t[\chi]}^{(L)}=-c_{i}^{(t+\Delta t / 2)} \sigma_{, i}^{(L)} \tag{7.16.15a}
\end{equation*}
$$

and

$$
\begin{align*}
\sigma_{, t t[\chi]}^{(L)} & =\left(\sigma_{, t[\chi]}^{(L)}\right)_{, t}=-\left(c_{i}^{(t+\Delta t / 2)} \sigma_{, i}^{(L)}\right)_{, t}=-c_{i}^{(t+\Delta t / 2)}\left(\sigma_{, t[\chi]}^{(L)}\right)_{i} \\
& =-c_{i}^{(t+\Delta t / 2)}\left(-c_{j}^{(t+\Delta t / 2)} \sigma_{, j}^{(L)}\right)_{, i}=c_{i}^{(t+\Delta t / 2)} c_{j}^{(t+\Delta t / 2)} \sigma_{, i j}^{(L)} \tag{7.16.15b}
\end{align*}
$$

After substituting the above two equations into the Taylor series expansion of $\sigma^{(t+\Delta t)}$ with respect to the time:

$$
\begin{equation*}
\sigma^{(t+\Delta t)}=\sigma^{t}+\sigma_{, t[\chi]}^{(L)} \Delta t+\frac{1}{2} \sigma_{, t t \chi \chi]}^{(L)} \Delta t^{2} \tag{7.16.16a}
\end{equation*}
$$

we obtain the update equation of the Lax-Wendroff method as:

$$
\begin{equation*}
\sigma^{(t+\Delta t)}=\sigma^{t}-c_{i}^{(t+\Delta t / 2)} \sigma_{i}^{(L)} \Delta t+\frac{1}{2} c_{i}^{(t+\Delta t / 2)} c_{j}^{(t+\Delta t / 2)} \sigma_{, i j}^{(L)} \Delta t^{2} \tag{7.16.16b}
\end{equation*}
$$

where $c_{i}^{(t+\Delta t / 2)}$ is the convective velocity evaluated at the mid-step.
In Eq. (7.16.16b), both the stress gradient, which will be denoted by $\gamma$, and its spatial derivatives are required. To obtain the gradient in Eq.(7.16.16b), a classical least-squares project is employed to obtain a smoothed field of stress gradient. Via divergence theorem, we have

$$
\begin{equation*}
\int_{\Omega} N^{\gamma} \gamma d \Omega=-\int_{\Omega} \sigma \nabla N^{\gamma} d \Omega+\int_{\Gamma} N^{\gamma} \sigma n d \Gamma \tag{7.16.17}
\end{equation*}
$$

where $n$ is the outward unit normal in the current configuration. After the regular assembling procedure, we obtain the linear set of equations:

$$
\begin{equation*}
\mathbf{M}^{\gamma} \gamma=\Phi \tag{7.16.18}
\end{equation*}
$$

where $\mathbf{M}^{\gamma}$ is a consistent pseudo-mass matrix defined as:

$$
\begin{equation*}
\mathbf{M}^{\gamma}=\left[M_{I J}^{\gamma}\right]=\int_{\Omega} N_{I}^{\gamma} N_{J}^{\gamma} d \Omega \tag{7.16.19a}
\end{equation*}
$$

$\gamma$ is the vector of nodal smoothed values of the stress gradient, and the vector $\Phi$ is defined as:

$$
\begin{equation*}
\Phi=\left[\Phi_{I}\right]=\sum_{e}\left[-\int_{\Omega} \sigma \nabla \mathbf{N}_{I}^{\gamma} d \Omega+\int_{\Gamma} \mathbf{N}_{I}^{\gamma} \sigma \mathbf{n} d \Gamma\right] \tag{7.16.19b}
\end{equation*}
$$

To make the algorithm explicit, the lumped mass matrix is preferred instead of consistent one. After doing so, the solution of stress gradient of $\gamma$ can be achieved straightforward. Then Eq.(7.16.16) can be solved. To obtain the stress value at Gauss points, the collocation technique can be applied to handle the weak form of Eq.(7.16.16).
(2) Godunov-like update:

In this phase, the convection equation of :

$$
\begin{equation*}
\sigma_{, t[\chi]}+c_{i} \sigma_{, i}=0 \tag{7.16.20}
\end{equation*}
$$

will be solved. With the help of the stress-velocity product $Y=\sigma c$, Eq.(7.16.20) can be rewritten as:

$$
\begin{equation*}
\sigma_{, t[\chi]}+Y_{i, i}=\sigma c_{i, i} \tag{7.16.21}
\end{equation*}
$$

To apply Godunov method, the weak form is presented

$$
\begin{equation*}
\int_{\Omega} \delta \sigma \sigma_{, t[\chi]} d \Omega=\int_{\Omega} \delta \sigma \sigma c_{i, i} d \Omega-\int_{\Gamma^{t}} \delta \sigma Y_{i} n_{i} d \Gamma \tag{7.16.22}
\end{equation*}
$$

where the test functions $\delta \sigma$ are constant within any single element. Since both $\sigma$ and $\delta \sigma$ are constants within an element $e$, Eq.(7.16.22) results in

$$
\begin{equation*}
\sigma_{, t[\chi]}=-\frac{1}{2 \Omega} \sum_{s=1}^{N_{s}} f_{s}\left(\sigma_{s}^{c}-\sigma\right)\left[1-\operatorname{sign}\left(f_{s}\right)\right] \tag{7.16.23}
\end{equation*}
$$

with the upwind consideration, where the element $e$ has volume $\Omega$ and $N_{s}$ faces, $\sigma_{s}^{c}$ is the stress component in the contiguous element across face $s$, and $f_{s}$ is the flux of convective velocity $c$ across face $s$,

$$
\begin{equation*}
f_{s}=\int_{s} c_{i} n_{i} d s \tag{7.16.24}
\end{equation*}
$$

To apply the Godunov update to the situation of multi-point quadrature, we can divide every finite element into various subelements, each of them corresponding to the influence domain of a Gauss point. In every subelement, $\sigma$ is assumed to be constant, and represented by the Gauss-point value. Because of this, $\sigma$ is a piece wise constant filed with respect to the mesh of subelements, and Eq.(7.16.24) can be integrated to update the value of $\sigma$ for each subelement.

### 7.16.3 Stress Update Procedures in 1-D Case:

In this section, we will compare the performance of different update procedure. Below, we will emphasize on the stress update. For illustrative purposes, we consider one-dimensional(1-D) case. In a 1-D case, the shape functions and the corresponding test functions for density, velocity, energy and stress-velocity product may be chosen to be the piecewise linear $C^{0}$ functions such that

$$
\begin{align*}
& N_{1}=\frac{1}{2}(1-\xi)  \tag{7.16.25a}\\
& N_{2}=\frac{1}{2}(1+\xi) \tag{7.16.25b}
\end{align*}
$$

where $\xi \in[-1,1]$, while the functions for deviatoric stress and pressure can be $C^{-1}$, or in particular, constant in each element. The test and trial functions for all variables are identical. The full upwind method can be applied for all the matrices involving convection effects.
For a uniform mesh when the 1-D rod is divided into M segments of equal size of h , where the elements and the nodes are numbered sequentially from 1 to $M$ and $M+1$, respectively. Let $c_{m}$ designate the convective velocity at node $m$ and $s_{m}$ the stress in element m . For simplicity, all the nodal convective velocities are considered to be positive.

### 7.16.3.1 Application of SUPG in 1-D Case:

The stress update is according to Eq.(7.16.7e). The matrices and vectors appeared in Eq.(7.16.7e) are as the following.

The generalized mass matrix is:

$$
\left.\mathbf{M}^{s}=h \left\lvert\, \begin{array}{ccccc}
\lceil 1 & & & &  \tag{7.16.26a}\\
\mid & \ddots & & & \\
\mid & & 1 & & \\
& & & \ddots & \\
& & & & 1
\end{array}\right.\right]_{M \times M}
$$

The transpose of the divergence operator matrix is

$$
\left.\mathbf{G}^{T}=\left\lvert\, \begin{array}{cccccc}
{[-1} & 1 & & & &  \tag{7.16.26b}\\
& \ddots & \ddots & & & \mid \\
& & -1 & 1 & & \mid \\
& & & \ddots & \ddots & \\
& & & & -1 & 1
\end{array}\right.\right]_{M \times(M+1)}
$$

The matrix $\mathbf{M}^{y}$ is diagonal by locating the integration points at the nodes

$$
\begin{equation*}
\operatorname{diag}\left(\mathbf{M}^{y}\right)=h\left\{\frac{1}{2}, 1, \cdots, 1, \cdots 1, \frac{1}{2}\right\}_{(M+1) \times 1}^{T} \tag{7.16.26c}
\end{equation*}
$$

The transport of stress vector is

$$
\begin{gather*}
\mathbf{L}^{y} \mathbf{s}=\frac{1}{6} h\left\{\left(2 c_{1}+c_{2}\right) s_{1}, \cdots,\left(c_{m-1}+2 c_{m}\right) s_{m-1}+\left(2 c_{m}+c_{m+1}\right) s_{m}, \cdots,\right. \\
\left.\left(c_{M}+2 c_{M+1}\right) s_{M}\right\}_{(M+1) \times 1}^{T} \tag{7.16.26d}
\end{gather*}
$$

if exact integration is used;
or

$$
\begin{equation*}
\mathbf{L}^{y} \mathbf{s}=\frac{1}{2} h\left\{0, \cdots,\left(c_{m}+c_{m+1}\right) s_{m}, \cdots,\left(c_{M}+c_{M+1}\right) s_{M}\right\}_{(M+1) \times 1}^{T} \tag{7.16.26e}
\end{equation*}
$$

if full upwind is used, where $1<m<M$ hereafter.

The generalized diffusion vector is

$$
\begin{equation*}
\mathbf{D s}=\left\{\left(-c_{1}+c_{2}\right) s_{1}, \cdots,\left(-c_{m}+c_{m+1}\right) s_{m}, \cdots,\left(-c_{M}+c_{M+1}\right) s_{M}\right\}_{M \times 1}^{T} \tag{7.16.26f}
\end{equation*}
$$

and the rate of change of stress due to material deformation ( the rotation of stress vanishes in 1-D case) is

$$
\begin{equation*}
\mathbf{z}=h\left\{\dot{s}_{1}, \cdots, \dot{s}_{m}, \cdots, \dot{s}_{M}\right\}_{M \times 1}^{T} \tag{7.16.26g}
\end{equation*}
$$

where $\dot{s}=C_{1111^{\prime}} v_{(1,1)}$
By substituting Eqs.(7.16.12a-g) into Eq.(7.16.7e), the rate of change of stress in ALE description can be shown to be:

$$
\left.\mathbf{s}_{t,[\chi]=}=\left\{\begin{array}{c}
s_{1,[\chi]} \\
\vdots \\
s_{m,[\chi]} \\
\vdots \\
s_{M,[\chi]}
\end{array}\right\}_{M \times 1}\left(\begin{array}{c}
\left(-c_{1}+c_{2}\right) s_{1} \\
\vdots \\
\left(-c_{m}+c_{m+1}\right) s_{m} \\
\vdots \\
\left(-c_{M}+c_{M+1}\right) s_{M}
\end{array}\right\}-\frac{1}{6 h}\left\{\begin{array}{c}
\left.\left(-3 c_{1}\right) s_{1}+\left(2 c_{2}+c_{3}\right) s_{2}\right) \\
\vdots \\
-\left(c_{m-1}+2 c_{m}\right) s_{m-1} \\
-\left(c_{m}-c_{m+1}\right) s_{m} \\
+\left(2 c_{m+1}+c_{m+2}\right) s_{m+1} \\
\vdots \\
-\left(c_{M-1}+2 c_{M}\right) s_{M-1} \\
+3 c_{M+1} s_{M}
\end{array}\right\}+\begin{array}{c} 
\\
\left\{\begin{array}{c}
s_{1,[\mathbf{x}]} \\
s_{m,[\mathbf{x}]} \\
\vdots
\end{array}\right\} \\
s_{M,[\mathbf{x}]}
\end{array}\right\}
$$

(7.16.27a)
if exact integration is used;
or

$$
\mathbf{s}_{t,[\chi]]}=\left\{\begin{array}{c}
s_{1,[\chi]}  \tag{7.16.27b}\\
\vdots \\
s_{m,[\chi]} \\
\vdots \\
s_{M,[\chi]}
\end{array}\right\}_{M \times 1}\left(\begin{array}{c}
\left(-c_{1}+c_{2}\right) s_{1} \\
\vdots \\
\left(-c_{m}+c_{m+1}\right) s_{m} \\
\vdots \\
\left(-c_{M}+c_{M+1}\right) s_{M}
\end{array}\right\}-\frac{1}{2 h}\left\{\begin{array}{c}
\left(-c_{1}+c_{2}\right) s_{1} \\
\vdots \\
\left(-c_{m-1}+c_{m}\right) s_{m-1} \\
+\left(c_{m}+c_{m+1}\right) s_{m} \\
\vdots \\
-\left(c_{M-1}+c_{M}\right) s_{M-1} \\
+2\left(c_{M}+c_{M+1}\right) s_{M}
\end{array}\right\}+\left\{\begin{array}{c}
\left.\left\lvert\, \begin{array}{c}
s_{1,[\mathbf{x}]} \\
\vdots \\
s_{m,[\mathbf{x}]} \\
\vdots \\
s_{M,[\mathbf{x}]}
\end{array}\right.\right\} \\
\mid
\end{array}\right\}
$$

if full upwind is used to evaluate $\mathbf{L}^{y} \mathbf{s}$.
The second bracket on the right-hand side of Eq.(7.16.27a) shows the central differencing (or simple averaging) effects for the transport of stresses, while Eq.(7.16.27b) exhibits the donor-cell differencing. This can be further clarified by letting

$$
c_{1}=\cdots=c_{m}=\cdots=c_{M+1}=c(\text { constant }),
$$

then

$$
\left.\left.\mathbf{s}_{t,[\chi]=}=\left\{\begin{array}{c}
s_{1,[\chi]}  \tag{7.16.28a}\\
\vdots \\
s_{m,[\chi]} \\
\vdots \\
s_{M,[\chi]}
\end{array}\right\}=-\frac{c}{2 h}\left\{-s_{m-1}+s_{m+1}\right\} \begin{array}{c}
-s_{1}+s_{2} \\
\vdots \\
\vdots \\
-s_{M-1}+s_{M}
\end{array}\right\}+\begin{array}{c}
\left.\left\lvert\, \begin{array}{c}
s_{1,[\mathbf{x}]} \\
\vdots \\
s_{m,[\mathbf{x}]} \\
\vdots \\
s_{M,[\mathbf{x}]}
\end{array}\right.\right\}
\end{array}\right\}
$$

if exact integration is used; and

$$
\mathbf{s}_{t,[\chi]=}\left\{\begin{array}{c}
s_{1,[\chi]}  \tag{7.16.28b}\\
\vdots \\
s_{m,[\chi]} \\
\vdots \\
\mid s_{M,[\chi]}
\end{array}\right\}_{M \times 1} \quad=-\frac{c}{h}\left\{\begin{array}{c}
s_{1} \\
\vdots \\
-s_{m-1}+s_{m} \\
\vdots \\
-s_{M-1}+2 s_{M}
\end{array}\right\}+\left\{\begin{array}{c}
s_{1,[\mathbf{x}]} \\
\vdots \\
s_{m,[\mathbf{x}]} \\
\vdots \\
s_{M,[\mathbf{x}]}
\end{array}\right\}
$$

if full upwind is used.
Eq.(7.16.28a) shows that the transport of the stresses at odd and even elements tends to be decoupled, therefore physically unrealistic oscillations would be expected when the simple averaging method is employed to evaluate the spatial derivatives of stresses.

### 7.16.3.2. Application of Operator Split in 1D Case:

Also, for illustrative purposes, a 1D rod is considered which is divided into M segments of equal size of $h$ and assume:

$$
c_{1}=\cdots=c_{m}=\cdots c_{M+1}=c(\text { constant })>0,
$$

## (1) Lax-Wendroff update:

We can see that this procedure of update will not work for constant stress and linear shape function since the RHS of Eq.(7.16.19) will be zero. In addition, we can see that for LaxWendroff update, the shape function of $\mathbf{s}, \mathbf{N}^{s}$, must be in the same order as $\mathbf{N}^{\gamma}$. Assuming both of them are linear shape functions, we can obtain:

$$
\begin{align*}
& \operatorname{diag}\left(\mathbf{M}^{y}\right)=h\left\{\frac{1}{2}, 1, \cdots, 1, \cdots 1, \frac{1}{2}\right\}_{(M+1) \times 1}^{T}  \tag{7.16.29a}\\
& \Phi=\frac{1}{2}\left[-s_{1}+s_{2},-s_{1}+s_{3},-s_{2}+s_{4}, \cdots,-s_{m-1}+s_{m+1}, \cdots-s_{M-1}+s_{M+1}\right]_{(M+1) \times 1}^{T} \tag{7.16.29b}
\end{align*}
$$

(2) Godunov-like update:

For constant stress and linear shape function, it is easy to obtain the update equation for element $n$

$$
\begin{equation*}
s_{n}^{(t+\Delta t)}=s_{n}^{(t)}-\frac{c \Delta t}{h}\left(s_{n}^{(t)}-s_{n-1}^{(t)}\right)+\Delta t \dot{s}_{n}^{(t+\Delta t / 2)} \tag{7.16.30}
\end{equation*}
$$

### 7.16.4 Explicit Time Integration Algorithm:

For simplicity, the coupled equations (7.16.7c-e) will be integrated by an explicit scheme. Lumped mass matrices are used to enhance the computational efficiency. Both predictorcorrector method( Hughes \& Liu, 1978) and standard central difference(Huerta \& Casadei, 1994) method can be applied here for explicit time integration. Below, ( $)_{n}$ and ( $)_{n+1}$ will denote the matrices at times $t_{n}=n \Delta t$ and $t_{n+1}=(n+1) \Delta t$ respectively, where $\Delta t$ is the time increment.
(1) Predictor-corrector method

This kind of predictor-corrector method is similar to the Newmark algorithm. The major difference is that the former algorithm is explicit, while the latter is implicit.

Mass equations:

$$
\begin{align*}
& \rho_{, t[\chi]_{n+1}}=-\left(\mathbf{M}_{n}^{\rho}\right)^{-1}\left(\mathbf{L}_{n}^{\rho} \tilde{\rho}_{n+1}+\mathbf{K}_{n}^{\rho} \tilde{\rho}_{n+1}\right)  \tag{7.16.31a}\\
& \tilde{\rho}_{n+1}=\rho_{n}+(1-\alpha) \Delta t \rho \rho_{, t[\chi]_{n}}  \tag{7.16.31b}\\
& \rho_{n+1}=\tilde{\rho}_{n+1}+\alpha \Delta t \rho_{, t[\chi]_{n+1}} \tag{7.16.31c}
\end{align*}
$$

Momentum equations:

$$
\begin{align*}
& \mathbf{v}_{, t[\chi]_{n+1}}=\left(\mathbf{M}_{n}\right)^{-1}\left(\mathbf{f}_{n+1}^{e x t}-\mathbf{f}_{n}^{i n t}-\mathbf{L}_{n} \tilde{\mathbf{v}}_{n+1}\right)  \tag{7.16.32a}\\
& \tilde{\mathbf{v}}_{n+1}=\mathbf{v}_{n}+(1-\gamma) \Delta t \mathbf{v}_{, t[\chi]_{n}}  \tag{7.16.32b}\\
& \mathbf{v}_{n+1}=\tilde{\mathbf{v}}_{n+1}+\gamma \Delta t \mathbf{v}_{, t[\chi]_{n+1}} \tag{7.16.32c}
\end{align*}
$$

Eq.(7.16.32a) needs to be used in conjunction with

$$
\begin{align*}
& \tilde{\mathbf{d}}_{n+1}=\mathbf{d}_{n}+\Delta t \mathbf{v}_{n}+\left(\frac{1}{2}-\beta\right) \Delta t^{2} \mathbf{v}_{, t[\chi]_{n}}  \tag{7.16.32d}\\
& \mathbf{d}_{n+1}=\tilde{\mathbf{d}}_{n+1}+\beta \Delta t^{2} \mathbf{v}_{, t[\chi]_{n+1}} \tag{7.16.32e}
\end{align*}
$$

to calculate the $\mathbf{f}_{n}^{\text {int }}$.
In the above equations, $\alpha, \beta$ and $\gamma$ are the computational parameters. For explicit calculations, the following constraints on the parameters are used:

$$
\begin{align*}
& \alpha=0  \tag{7.16.33a}\\
& \beta=0 \tag{7.16.33b}
\end{align*}
$$

$$
\begin{equation*}
\gamma \geq \frac{1}{2} \tag{7.16.33c}
\end{equation*}
$$

The flowchart of the computational procedure for the class of pressure-insensitivematerials is as follows:
Step 1. Initialization. Set $n=0$, input initial conditions.
Step 2. Time stepping loop, $t \in\left[0, t_{\max }\right]$.
Step 3. Integrate the mesh velocity to obtain the mesh displacement and spatial coordinates. Step 4. Calculate incremental hydrostatic pressure by integration Eq.(7.16.7e) with $s$ and $z$ replaced by $\boldsymbol{p}$ and $\boldsymbol{u}$ respectively:
(a) the rate of pressure due to convection,
(b) the rate of pressure due to deformation.

Step 5. Calculate incremental deviatoric stresses, yield stresses, and back stresses by integration Eq.(7.16.7e) which stress update procedures have been discussed in detail in last section:
(a) the rate of stresses due to convection,
(b) the rate of stresses due to rotation,
(c) the rate of stresses due to deformation.

Step 6. Compute the internal force vector.
Step 7. Compute the acceleration by the equations of motion, Eq.(7.16.32a).
Step 8. Compute the density by the equation of mass conservation, Eq.(7.16.31a).
Step.7. Integrate the acceleration to obtain the velocity.
Step 10. If $(n+1) \Delta t>t_{\max }$, stop; otherwise, replace $n$ by $n+l$ and go to Step 2

## (2) Central difference method

The central difference can also be applied for the time integration. Same as the standard central difference procedure, we can obtain displacement and acceleration vectors at each $i$ time step, while get velocity vector at each $i+\frac{1}{2}$ time step. The momentum equation will be integrated as an example to illustrate this explicit scheme.

After obtaining the velocity vector at $\left(n-\frac{1}{2}\right) \Delta t$ and the displacement vector at $n \Delta t$, the acceleration vector at $n \Delta t$ is :

$$
\mathbf{v}_{, t[\chi]_{n}}=\left(\mathbf{M}_{n}\right)^{-1}\left(\mathbf{f}_{n}^{e x t}-\mathbf{f}_{n}^{i n t}-\mathbf{L}_{n-\frac{1}{2}} \mathbf{v}_{n-\frac{1}{2}}\right)
$$

With central difference scheme, the velocity and displacement of next time step are:

$$
\mathbf{v}_{n+\frac{1}{2}}=\mathbf{v}_{n-\frac{1}{2}}+\Delta t \mathbf{v}_{t[\chi \chi]_{n}}
$$

and

$$
\mathbf{d}_{n+1}=\mathbf{d}_{n}+\Delta t \mathbf{v}_{n+\frac{1}{2}}
$$

For operator splitting update, the Lagrangian part for strains is calculated with a usual central difference scheme as:

$$
\varepsilon\left(t^{*}\right)=\varepsilon_{n}+\Delta t \varepsilon_{, t[\mathbf{x}]_{n+\frac{1}{2}}}^{(L)}
$$

where

$$
\varepsilon_{, t[\mathbf{x}]_{n+\frac{1}{2}}}^{(L)}=\left(\mathbf{v}_{, x}\right)_{n+\frac{1}{2}}
$$

After considering the Godunov scheme for the convect part, the full upwind integration for 1D 2-node element can be written as:

$$
\varepsilon_{n+1}=\varepsilon\left(t^{*}\right)-\frac{\Delta t}{h}\left[c_{1} \frac{1+\operatorname{sign}\left(c_{1}\right)}{2}\left(\varepsilon_{e}-\varepsilon_{e-1}\right)+c_{2} \frac{1-\operatorname{sign}\left(c_{2}\right)}{2}\left(\varepsilon_{e+1}-\varepsilon_{e}\right)\right]
$$

For elastic-plastic problem, the radial return method can be used to determine the correct states of stress and strain.

### 7.10 ALE Governing Equation

7.10.1 Slightly Compressible Viscous Flow with Moving Boundary Problem

In this section, we develop the governing equations for a slightly compressible Newtonian fluid. In a generalized Newtonian fluid, the stress is a function of the rate-ofdeformation. Therefore, the stress is independent of the history of deformation. The most well known case is a linear Newtonian fluid, where the stress is a linear function of the rate-of-deformation.

This class of materials simplifies the implementation since the constitutive equation is independent of the strain history. Therefore, the constitutive equation can be used in its strong form.

The formulation is restricted to isothermal processes; therefore energy equation is not needed. The continuity equation (7.7.15), momentum equation (7.7.25) and the mesh update equation (7.2.19) in ALE description are:

$$
\begin{align*}
& \left.\frac{\partial \rho}{\partial t}\right|_{\chi}+c_{i} \frac{\partial \rho}{\partial x_{i}}+\rho \frac{\partial v_{i}}{\partial x_{i}}=0  \tag{7.10.1a}\\
& \left.\rho \frac{\partial v_{i}}{\partial t}\right|_{\chi}+\rho c_{j} \frac{\partial v_{i}}{\partial x_{j}}=\frac{\partial \sigma_{i j}}{\partial x_{j}}+\rho b_{i}  \tag{7.10.1b}\\
& \frac{\partial x_{i}}{\partial \chi_{j}} w_{j}=c_{i} \tag{7.10.1c}
\end{align*}
$$

In the constitutive equation, the hydrostatic stress is independent of deformation while the shear stress, which depend on the rate of deformation.

$$
\begin{equation*}
\sigma_{i j}=-p \delta_{i j}+2 \mu D_{i j} \tag{7.10.1d}
\end{equation*}
$$

where

$$
\begin{equation*}
D_{i j}=\frac{1}{2}\left(\frac{\partial v_{i}}{\partial x_{j}}+\frac{\partial v_{j}}{\partial x_{i}}\right) \text { and } \mu=\mu\left(D_{i j}\right) \tag{7.10.1e}
\end{equation*}
$$

where $\mu$ is the dynamic viscosity which is shear rate dependent. The functions $\mu$ for generalized Newtonian models are presented in Table 7.2.

| Model | 1-D Viscosity | 3-D Generalization |
| :---: | :---: | :---: |
| Newtonian | $\mu_{0}=$ constant | $\mathbf{s}=2 \mu_{0} \mathbf{D}$ |
| Power Law | $\mu=m D^{n-1}$ | $\mathbf{s}=2 m\left[\sqrt{2 \operatorname{tr}(\mathbf{D})^{2}}\right]^{n-1} \mathbf{D}$ |
| Truncated Power Law | $\begin{aligned} & \text { if } \quad D \leq D_{0} \\ & \quad \mu=\mu_{0} \\ & \text { if } \quad D \geq D_{0} \\ & \\ & \quad \mu=\mu_{0}\left(\frac{D}{D_{0}}\right)^{n-1} \end{aligned}$ | if $\sqrt{2 \operatorname{tr}(\mathbf{D})^{2}} \geq D_{0}$ : $\mathbf{s}=2 \mu_{0}\left[\frac{\sqrt{2 \operatorname{tr}(\mathbf{D})^{2}}}{\dot{\gamma}_{0}}\right]^{n-1} \mathbf{D}$ |
| Carreau | $\frac{\mu-\mu_{\infty}}{\mu_{0}-\mu_{\infty}}=\left[1+(\lambda D)^{2}\right]^{(n-1) / 2}$ | $\begin{aligned} & \mathbf{s}=2 \mu_{\infty} \mathbf{D} \\ & \quad+2\left(\mu_{0}-\mu_{\infty}\right)\left[1+2 \lambda^{2} \operatorname{tr}(\mathbf{D})^{2}\right]^{(n-1) / 2} \mathbf{D} \end{aligned}$ |
| Carreau-A | $\mu_{\infty}=0$ | $\mathbf{s}=2 \mu_{0}\left[1+2 \lambda^{2} \operatorname{tr}(\mathbf{D})^{2}\right]^{(n-1) / 2} \mathbf{D}$ |
| Bingham | $\begin{aligned} & \mu=\infty \quad \tau \leq \tau_{0} \\ & \mu=\mu_{p}+\frac{\tau_{0}}{D} \quad \tau \geq \tau_{0} \end{aligned}$ | $\begin{aligned} & \text { if } \left.\begin{array}{l} \frac{1}{2} \operatorname{tr}(\mathbf{s})^{2} \leq \tau_{0}^{2} \\ \quad \mathbf{s}=\mathbf{0} \\ \text { if } \frac{1}{2} \operatorname{tr}(\mathbf{s})^{2} \geq \tau_{0}^{2} \\ \quad \mathbf{s}=2 \mu_{p}\left[1+\frac{\tau_{0}}{\sqrt{2 \operatorname{tr}(\mathbf{D})^{2}}}\right] \end{array}\right] \end{aligned}$ |

where $\mathbf{s}$ and $\mathbf{D}$ are the deviatoric part of the stress and stretch tensors, respectively

We will first consider a viscous, barotropic fluid, so that the pressure depends only on the density, $p=p(\rho)$. The material time derivative of the pressure $p$ gives:

$$
\begin{equation*}
\dot{p}=\frac{\partial p}{\partial \rho} \dot{\rho} \tag{7.10.2a}
\end{equation*}
$$

Now we can define the bulk modules $B$ by:

$$
\begin{equation*}
\frac{B}{\rho}=\frac{\partial p}{\partial \rho} \tag{7.10.2b}
\end{equation*}
$$

Using this definition in Eq. (7.10.2a) yields:

$$
\begin{equation*}
\dot{p}=\frac{B}{\rho} \dot{\rho} \text { or } \frac{\dot{\rho}}{\rho}=\frac{\dot{p}}{B} \tag{7.10.2c}
\end{equation*}
$$

Substituting Eq. (7.10.2c) into Eq. (7.10.1a), the continuity equation may be rewritten as (Liu and Ma, 1982):

$$
\begin{equation*}
\frac{1}{B} \frac{D p}{D t}+\frac{\partial v_{i}}{\partial x_{i}}=0 \tag{7.10.3a}
\end{equation*}
$$

or, by introducing Eq. (7.2.20b) in (7.10.3a), as:

$$
\begin{equation*}
\left.\frac{1}{B} \frac{\partial p}{\partial t}\right|_{\chi}+\frac{1}{B} c_{i} \frac{\partial p}{\partial x_{i}}+\frac{\partial v_{i}}{\partial x_{i}}=0 \tag{7.10.3b}
\end{equation*}
$$

The objective here is to find the density, the material velocity and the mesh velocity by solving Eqs. (7.10.1a-e). Prior to presenting the strong from of the governing equations and boundary conditions, the finite element mesh updating procedure will first be discussed in the next section.

In each Eq (7.10.1) a convective term is present; thus, one of the drawbacks of an Eulerian formulation is retained in the ALE methods. The major advantage of an ALE approach is that it simplifies the treatment moving boundaries and interfaces.

### 7.11 Mesh Update Equations

### 7.11.1 Introduction

The option of arbitrarily moving the mesh in the ALE description offers interesting possibilities. By means of ALE, moving boundaries (which are material surfaces) can be tracked with the accuracy characteristic of Lagrangian methods and the interior mesh can be moved so as to avoid excessive element distortion and entanglement. However, this requires that an effective algorithm for updating the mesh, i.e. the mesh velocities $\hat{\mathbf{v}}$, must
be prescribed. The mesh should be prescribed so that mesh distortion is avoided and so that boundaries and interfaces remain at least partially Lagrangian.

In this section, we will describe several procedures for updating the mesh. The material and mesh velocities are related by Eq. (7.2.19); hence, once one of them is determined, the other is automatically fixed. It is important to note that, if $\hat{\mathbf{v}}$ is given, $\hat{\mathbf{d}}$ and $\hat{\mathbf{a}}$ can be computed and there is no need to evaluate $\mathbf{w}$. On the other hand, if $\hat{\mathbf{v}}$ is considered the unknown but $\mathbf{w}$ is given, Eq. (7.2.19) must be solved to evaluate $\hat{\mathbf{v}}$ before updating the mesh. Finally, mixed reference velocities can be given (i.e. a component of $\hat{\mathbf{v}}$ can be prescribed and $\mathbf{w}$ in the other(s)). Finding the best choice for these velocities and an algorithm for updating the mesh constitutes one of the major hurdles in developing an effective implementation the ALE description. Depending on which velocity ( $\hat{\mathbf{v}}$ or $\mathbf{w}$ or mixed) is prescribed, three different cases may be studied.

### 7.11.2 Mesh Motion Prescribed a Priori

The case where the mesh motion $\hat{\mathbf{v}}$ is given corresponds to an analysis where the domain boundaries are known at every instant. When the boundaries of the fluid domain have a known motion, the mesh movement along this boundary can be prescribed a priori.

### 7.11.3 Lagrange-Euler Matrix Method

The case where the relative velocity $\mathbf{w}$ is arbitrarily defined is a format for apropos by Hughes et al (1981). Let w be:

$$
\begin{equation*}
w_{i}=\left.\frac{\partial \chi_{i}}{\partial t}\right|_{X}=\left(\delta_{i j}-\alpha_{i j}\right) v_{j} \tag{7.11.1}
\end{equation*}
$$

where $\delta_{i j}$ is the Kroneker delta and $\alpha_{i j}$ is the Lagrange-Euler parameter matrix such that $\alpha_{i j}=0$ if $i \neq j$ and $\alpha_{i j}$ is real (underlined indices meaning no sum on them). In general, the $\alpha^{\prime}$ s can vary in space and be time-dependent; however $\alpha_{i j}$ is usually taken as timeindependent. According to Eq. (7.11.1) the relative velocity $\mathbf{w}$ becomes a linear function of the material velocity and it was chosen because, if $\alpha_{i j}=\delta_{i j}, \mathbf{w}=\mathbf{0}$ and the Lagrangian description is obtained, whereas, if $\alpha_{i j}=0, \mathbf{w}=\mathbf{v}$ and the Eulerian formulation is used. The Lagrange-Euler matrix needs to be given once and for all at each grid point.

Since $\mathbf{w}$ is defined by Eq. (7.11.1), the other velocities are determined by Eq. (7.2.19), which become, respectively.

$$
\begin{equation*}
c_{i}=\frac{\partial x_{i}}{\partial \chi_{j}}\left(\delta_{j k}-\alpha_{j k}\right) v_{k} \tag{7.11.2a}
\end{equation*}
$$

and

$$
\begin{equation*}
\hat{v}_{i}=v_{i}-\left(\delta_{j k}-\alpha_{j k}\right) v_{k} \frac{\partial x_{i}}{\partial \chi_{j}} \tag{7.11.2b}
\end{equation*}
$$

The latter equations must be satisfied in the referential domain along its boundaries. Substituting Eq. (7.2.8a) into (7.11.2b) yields a basic equation for mesh rezoning:

$$
\begin{equation*}
\left.\frac{\partial x_{i}}{\partial t}\right|_{\chi}+\left(\delta_{j k}-\alpha_{j k}\right) v_{k} \frac{\partial x_{i}}{\partial \chi_{j}}-v_{i}=0 \tag{7.11.3}
\end{equation*}
$$

The explicit form of Eq. (7.11.3) in 1D, 2D and 3D are listed:

## 1D Form

$$
\begin{equation*}
\left.\frac{\partial x}{\partial t}\right|_{\chi}+(1-\alpha) v \frac{\partial x}{\partial \chi}-v=0 \tag{7.11.4}
\end{equation*}
$$

2D Form

$$
\begin{align*}
& \left.\frac{\partial x_{1}}{\partial t}\right|_{\chi}+\left(1-\alpha_{11}\right) v_{1} \frac{\partial x_{1}}{\partial \chi_{1}}+\left(1-\alpha_{22}\right) v_{2} \frac{\partial x_{1}}{\partial \chi_{2}}-v_{1}=0  \tag{7.11.5a}\\
& \left.\frac{\partial x_{2}}{\partial t}\right|_{\chi}+\left(1-\alpha_{11}\right) v_{1} \frac{\partial x_{2}}{\partial \chi_{1}}+\left(1-\alpha_{22}\right) v_{2} \frac{\partial x_{2}}{\partial \chi_{2}}-v_{2}=0 \tag{7.11.5b}
\end{align*}
$$

3D Form

$$
\begin{align*}
& \left.\frac{\partial x_{1}}{\partial t}\right|_{\chi}+\left(1-\alpha_{11}\right) v_{1} \frac{\partial x_{1}}{\partial \chi_{1}}+\left(1-\alpha_{22}\right) v_{2} \frac{\partial x_{1}}{\partial \chi_{2}}+\left(1-\alpha_{33}\right) v_{3} \frac{\partial x_{1}}{\partial \chi_{3}}-v_{1}=0  \tag{7.11.6a}\\
& \left.\frac{\partial x_{2}}{\partial t}\right|_{\chi}+\left(1-\alpha_{11}\right) v_{1} \frac{\partial x_{2}}{\partial \chi_{1}}+\left(1-\alpha_{22}\right) v_{2} \frac{\partial x_{2}}{\partial \chi_{2}}+\left(1-\alpha_{33}\right) v_{3} \frac{\partial x_{2}}{\partial \chi_{3}}-v_{2}=0  \tag{7.11.6b}\\
& \left.\frac{\partial x_{3}}{\partial t}\right|_{\chi}+\left(1-\alpha_{11}\right) v_{1} \frac{\partial x_{3}}{\partial \chi_{1}}+\left(1-\alpha_{22}\right) v_{2} \frac{\partial x_{3}}{\partial \chi_{2}}+\left(1-\alpha_{33}\right) v_{3} \frac{\partial x_{3}}{\partial \chi_{3}}-v_{3}=0 \tag{7.11.6c}
\end{align*}
$$

## Remark 1:

Equation (7.11.3) differs only in its last term from the one proposed by Hughes et al (1981). This difference is not noticeable if the Lagrange-Euler parameters $\alpha_{i j}$ are chosen equal to zero or one. Moreover, Eq. (7.11.3) includes the Jacobian matrix (i.e. $\partial x_{i} / \partial \chi_{j}$ ) that is missing in the Liu and Ma (1982) formulation. Finally, Eq. (7.11.3) is a transport equation without any diffusion so the classic numerical difficulties associated with transport equations are expected.

The ALE technique with a mesh update based on the Lagrange-Euler parameters is very useful in surface wave problems. We assume that the free surface is oriented relative to the global coordinates so that it can be written as $x_{3 s}=x_{3 s}\left(x_{1}, x_{2}, t\right)$. An Eulerian description used in the $x_{1}$ and $x_{2}$ directions (i.e. $x_{1}=\chi_{1}$ and $x_{2}=\chi_{2}$ ). The free surface is defined by one spatial coordinate which is a continuous and differentiable function of the other two spatial coordinates and time. In this case the Lagrange Euler matrix has only one non-zero term, $\alpha_{33}$ (usually equal to 1 ), and the only non-trivial equation in (7.11.3) is:

$$
\begin{equation*}
\left.\frac{\partial x_{3 s}}{\partial t}\right|_{\chi}+v_{1} \frac{\partial x_{3 s}}{\partial \chi_{1}}+v_{2} \frac{\partial x_{3 s}}{\partial \chi_{2}}-v_{3}=\left(\alpha_{33}-1\right) v_{3} \frac{\partial x_{3 s}}{\partial \chi_{3}} \tag{7.11.7}
\end{equation*}
$$

The above equation is easily recognized as the kinematics equation of the surface and may be written as:

$$
\begin{equation*}
\left.\frac{\partial x_{3 s}}{\partial t}\right|_{\chi}+v_{i} n_{i} N_{s}=a\left(x_{1}, x_{2}, x_{3 s}, t\right) \tag{7.11.8}
\end{equation*}
$$

where the components of $\mathbf{n}$ is the unit normal pointing out from the surface. The components of the normal vector are given by:

$$
\begin{equation*}
\frac{1}{N_{s}}\left(\frac{\partial x_{3 s}}{\partial \chi_{1}}, \frac{\partial x_{3 s}}{\partial \chi_{2}},-1\right) \tag{7.11.9a}
\end{equation*}
$$

with $N_{s}$ given by:

$$
\begin{equation*}
N_{s}=\left[1+\left(\frac{\partial x_{3 s}}{\partial \chi_{1}}\right)^{2}+\left(\frac{\partial x_{3 s}}{\partial \chi_{2}}\right)^{2}\right\rceil^{1 / 2}=\left[1+\left(\frac{\partial x_{3 s}}{\partial x_{1}}\right)^{2}+\left(\frac{\partial x_{3 s}}{\partial x_{2}}\right)^{2}\right]^{1 / 2} \tag{7.11.9b}
\end{equation*}
$$

where $a\left(x_{1}, x_{2}, x_{3 s}, t\right)$ is the so-called accumulation rate function expressing the gain or loss of mass under the free surface (Hutter and Vulliet, 1985). It can be seen by comparing Eq. (7.11.7) and Eq. (7.11.8) that the accumulation rate function is:

$$
\begin{equation*}
a\left(x_{1}, x_{2}, x_{3 s}, t\right)=\left(\alpha_{33}-1\right) v_{3} \frac{\partial x_{3 s}}{\partial \chi_{3}}=w_{3} \frac{\partial x_{3 s}}{\partial \chi_{2}} \tag{7.11.10}
\end{equation*}
$$

The free surface is a material surface; along the free surface the accumulation rate must be zero, and consequently $\alpha_{33}$ has to be taken equal to one. This can also be deduced by noticing that no particles can cross the free surface, so $w_{3}$ must be zero. Although Eq. (7.11.3) can be applied to problems where $x_{1}$ and/or $x_{2}$ are not Eulerian by prescribing non zero $\alpha$ 's in these directions, controlling the element shapes by adjusting the $\alpha$ 's is very difficult.

Control of the mesh by Eq. (7.11.1) has some disadvantages; for instance, while $\hat{\mathbf{v}}$ has a clear physical interpretation (i.e. the mesh velocity), $\mathbf{w}$ is much more difficult to visualize (except in the direction perpendicular to material surfaces, where it is identically zero) and therefore it is very difficult to maintain regular shaped elements inside the fluid domain by just prescribing the $\alpha$ 's. Because of this important drawback the mixed formulation introduced by Huerta and Liu, called deformation gradient method, was developed, it is discussed next.

### 7.11.4 Deformation Gradient Formulations

Noticing the restrictions of the $\alpha$ 's scheme, a mixed formulation is developed for the resolution of Eq. (7.2.19). One of the goals of the ALE method is the accurate tracking of the moving boundaries which are usually material surfaces. Hence, along these surfaces we enforce $\mathbf{w} \cdot \mathbf{n}=0$ where $\mathbf{n}$ is the exterior normal. The other goal of the ALE technique is to avoid element entanglement and this is better achieved, once the boundaries are known, by prescribing the mesh displacements independently (through the potential equations, for instance) or velocities, because both $\hat{\mathbf{d}}$ and $\hat{\mathbf{v}}$ govern directly the element shape. Therefore, one can prescribe $\mathbf{w} \cdot \mathbf{n}=0$ along the domain boundaries while defining the $\hat{\mathbf{d}}$ 's or $\hat{\mathbf{v}}$ 's in the interior.

The system of differential equations defined in Eq. (7.2.19) has to be solved along the moving boundaries. Notice first that solving for $w_{i}$ in terms of $\left(v_{i}-\hat{v}_{i}\right)$, Eq. (7.2.19) can be rewritten as:

$$
\begin{equation*}
c_{j} \equiv v_{j}-\hat{v}_{j}=F_{j i}^{\chi} w_{i} \tag{7.11.11}
\end{equation*}
$$

Define the Jacobian matrix of the map between the spatial and ALE coordinates by

$$
\begin{equation*}
F_{i j}^{\chi} \equiv \frac{\partial x_{i}}{\partial \chi_{j}} \tag{7.11.12}
\end{equation*}
$$

Its inverse is:

$$
\left.\left(\mathbf{F}^{\chi}\right)^{-1}=\frac{1}{\hat{J}} \left\lvert\, \begin{array}{ccc}
\hat{J}_{11} & -\hat{J}_{12} & \hat{J}_{13}  \tag{7.11.13a}\\
-\hat{J}_{21} & \hat{J}_{22} & -\hat{J}_{23} \\
\hat{J}_{31} & -\hat{J}_{32} & \hat{J}_{33}
\end{array}\right.\right] \equiv \frac{\hat{J}_{i j}}{\hat{J}}
$$

where $\hat{J}_{i j}$ are the cofactors of $F_{i j}^{\chi}, \hat{J}$ is the Jacobian already defined in Eq. (7.7.4b) and $\hat{J}^{i j}$ are the minors of the Jacobian matrix $F_{i j}^{\chi}$. Multiplying the inverse Jacobian matrix on both sides of Eq. (7.11.11) and substituting Eq. (7.11.13) into Eq. (7.11.11), yields:

$$
\begin{equation*}
\frac{\hat{J}_{i j}}{\hat{J}} v_{j}-\hat{v}_{j}=w_{i} \text { or } \hat{J}^{j i}\left(v_{j}-\hat{v}_{j}\right)=\hat{J} w_{i} \tag{7.11.14}
\end{equation*}
$$

Dividing $\hat{J}^{i i}$ on both sides of Eqs. (7.11.14), gives:

$$
\frac{\hat{J}^{j i}}{\hat{J}^{i \underline{i}}}\left(v_{j}-\hat{v}_{j}\right)=\frac{\hat{J}}{\hat{J}^{i \underline{i}}} w_{i}=\left\{\begin{array}{l}
\left.\frac{\partial x_{i}}{\partial t}\right|_{\chi}-v_{i} \quad i=j  \tag{7.11.15}\\
\sum_{\substack{j=1 \\
j \neq i}}^{n s d} \frac{v_{j}-\hat{v}_{j}}{\hat{J}^{\underline{i}}} \hat{J}^{j \underline{i}} \quad i \neq j
\end{array}\right.
$$

When the LHS of Eq. (7.11.15) has been simplified using by the definition of $\hat{v}_{j}$, Eq. (7.2.8a). Substituting Eq. (7.11.16) into Eq. (7.11.15) yields:

$$
\begin{equation*}
\left.\frac{\partial x_{i}}{\partial t}\right|_{\chi}-v_{i}-\sum_{\substack{j=1 \\ j \neq i}}^{N S D} \frac{v_{j}-\hat{v}_{j}}{\hat{J}^{\underline{i}}} \hat{J}^{j \underline{i}}=-\frac{\hat{J}}{\hat{J}^{\underline{i}}} w_{i} \tag{7.11.17}
\end{equation*}
$$

Notice that the cofactor $\hat{J}^{\underline{i}}$ appears in the denominator to account for the motion of the mesh in the plane perpendicular to $\chi_{i}$ because Eqs. (7.11.17) are verified in the reference domain $\hat{\Omega}$, not in the actual deformed domain $\Omega$.

Examples for Eq. (7.11.17) in 1D, 2D and 3D are:

1D

$$
\begin{equation*}
\left.\frac{\partial x_{1}}{\partial t}\right|_{\chi}-v_{1}=-\frac{\hat{J}}{\hat{J}^{11}} w_{1} \tag{7.11.18}
\end{equation*}
$$

where $\hat{J}^{11}=1$.
2D

$$
\begin{align*}
& \left.\frac{\partial x_{1}}{\partial t}\right|_{\chi}-v_{1}-\frac{v_{2}-\hat{v}_{2}}{\hat{J}^{11}} \hat{J}^{21}=-\frac{\hat{J}}{\hat{J}^{11}} w_{1}  \tag{7.11.19a}\\
& \left.\frac{\partial x_{2}}{\partial t}\right|_{\chi}-v_{2}-\frac{v_{1}-\hat{v}_{1}}{\hat{J}^{22}} \hat{J}^{12}=-\frac{\hat{J}}{\hat{J}^{22}} w_{2} \tag{7.11.19b}
\end{align*}
$$

where $\hat{J}^{11}=\frac{\partial x_{2}}{\partial \chi_{2}}$ and $\hat{J}^{22}=\frac{\partial x_{1}}{\partial \chi_{1}}$.
3D

$$
\begin{align*}
& \left.\frac{\partial x_{1}}{\partial t}\right|_{\chi}-v_{1}-\frac{v_{2}-\hat{v}_{2}}{\hat{J}^{11}} \hat{J}^{21}-\frac{v_{3}-\hat{v}_{3}}{\hat{J}^{11}} \hat{J}^{31}=-\frac{\hat{J}}{\hat{J}^{11}} w_{1}  \tag{7.11.20a}\\
& \left.\frac{\partial x_{2}}{\partial t}\right|_{\chi}-v_{2}-\frac{v_{1}-\hat{v}_{1}}{\hat{J}^{22}} \hat{J}^{12}-\frac{v_{3}-\hat{v}_{3}}{\hat{J}^{22}} \hat{J}^{32}=-\frac{\hat{J}}{\hat{J}^{22}} w_{2}  \tag{7.11.20b}\\
& \left.\frac{\partial x_{3}}{\partial t}\right|_{\chi}-v_{3}-\frac{v_{1}-\hat{v}_{1}}{\hat{J}^{33}} \hat{J}^{13}-\frac{v_{2}-\hat{v}_{2}}{\hat{J}^{33}} \hat{J}^{23}=-\frac{\hat{J}}{\hat{J}^{33}} w_{3} \tag{7.11.20c}
\end{align*}
$$

For purposes of simplification and without any loss of generality, assume that the moving boundaries are perpendicular to one coordinate axis in the reference domain. Let the free surface be perpendicular to $\chi_{3}$, the first two equations in Eq. (7.11.17) are trivial because in the direction of $\chi_{1}$ and $\chi_{2}$ the mesh velocity is prescribed and therefore the mesh motion is known, but the third one must be solved for $\hat{v}_{3}$ given $w_{3}, \hat{v}_{1}$, and $\hat{v}_{2}$, it may be written explicitly as:

$$
\begin{equation*}
\hat{v}_{3}-\frac{\hat{J}^{13}}{\hat{J}^{33}}\left(v_{1}-\hat{v}_{1}\right)-\frac{\hat{J}^{23}}{\hat{J}^{33}}\left(v_{2}-\hat{v}_{2}\right)-v_{3}=-\frac{\hat{J}}{\hat{J}^{33}} w_{3} \tag{7.11.21a}
\end{equation*}
$$

or

$$
\begin{align*}
& \left.\frac{\partial x_{3 s}}{\partial t}\right|_{\chi}-\frac{v_{1}-\hat{v}_{1}}{\hat{J}^{33}} \hat{J}^{13}\left(\frac{\partial x_{3 s}}{\partial \chi_{1}}, \frac{\partial x_{3 s}}{\partial \chi_{2}}\right)-\frac{v_{2}-\hat{v}_{2}}{\hat{J}^{33}} \hat{J}^{23}\left(\frac{\partial x_{3 s}}{\partial \chi_{1}}, \frac{\partial x_{3 s}}{\partial \chi_{2}}\right)-v_{3} \\
& =-\frac{w_{3}}{\hat{J}^{33}} \hat{J}\left(\frac{\partial x_{3 s}}{\partial \chi_{1}}, \frac{\partial x_{3 s}}{\partial \chi_{2}}\right) \tag{7.11.22b}
\end{align*}
$$

where $\hat{v}_{3}$ has been substituted by $\left.\frac{\partial x_{3 s}}{\partial t}\right|_{\chi}, \hat{J}^{13}, \hat{J}^{23}$, and the Jacobian $\hat{J}$ are function of $\frac{\partial x_{3 s}}{\partial \chi_{1}}$ and $\frac{\partial x_{3 s}}{\partial \chi_{2}}: \hat{J}^{33}$ is not dependent on $x_{3 s}$ : and $x_{3 s}$ is the free surface equation. In Eq. (7.11.22b) $x_{3 s}$ is the unknown function, while $\hat{v}_{1}, \hat{v}_{2}$ and $w_{3}$ are known. If $\hat{v}_{1}=\hat{v}_{2}=0$ (i.e. the Eulerian description is used in $\chi_{1}$ and $\chi_{2}$ ), the kinematic surface equation, Eq. (7.11.8), is again obtained. However, with the mixed formulation $\hat{v}_{1}$ and $\hat{v}_{2}$ can be prescribed (as a percentage of the wave celerity, for instance) and therefore betternumerical results are obtained than by defining in Eq. (7.11.7) $\alpha_{11}$ and $\alpha_{22}$, whose physical interpretation is much more obscure.

### 7.11.5 Automatic Mesh Generation

The Laplacian method for remeshing is based on mapping the new position of the nodes by solutions of the Laplace equation space $(I, J)$ into real space $(x, y)$ through solving the Laplace differential equation is the most commonly use one.

The determination of the nodes is posed as finding $x(I, J)$ and $y(I, J)$, such that they satisfy the following equations

$$
\begin{equation*}
L^{2}(x)=\frac{\partial^{2} x}{\partial I^{2}}+\frac{\partial^{2} x}{\partial J^{2}}=0 ; L^{2}(y)=\frac{\partial^{2} y}{\partial I^{2}}+\frac{\partial^{2} y}{\partial J^{2}}=0 \quad \text { in } \Omega_{x} \tag{7.11.23a}
\end{equation*}
$$

The boundary conditions in two dimension are

$$
\begin{equation*}
x(I, J)=\bar{x}(I, J) ; y(I, J)=\bar{y}(I, J) \quad \text { in } \Gamma_{x} \tag{7.11.23b}
\end{equation*}
$$

Here $x(I)$ and $y(J)$ are the coordinates of nodes $I$ and $J$ when $I$ and $J$ take on integer values. $\Omega$ and $\Gamma$ are the domain and boundary of the remesh region and $L^{2}$ is the second-order differential operator.

Another useful mesh generation scheme is by solving a fourth order differential equation

$$
\begin{equation*}
L^{4}(x)=\frac{\partial^{4} x}{\partial I^{2} \partial J^{2}} ; L^{4}(y)=\frac{\partial^{4} x}{\partial I^{2} \partial J^{2}} \tag{7.11.24}
\end{equation*}
$$

Eqs. (7.11.23) and (7.11.24) can be solved by the finite difference method with a GaussSeidel iteration scheme. Meshes generated by the Laplace equation are distorted near the boundary where a high curvature occurs. However, the fourth-order equation gives a better mesh shape, because a higher differentiation is employed. An equipotential method regards the mesh lines as two intersecting sets of equipotentials, with each set satisfying Laplace's equation in the interior with adequate boundary condition.

### 7.12 Strong Form, Governing Equations of Slightly Compressible Viscous Flow with Moving Boundary Problem

The object here is to find the following functions:

$$
\begin{equation*}
v(x, t)=\text { velocity }, p(x, t)=\text { pressure fields, } \hat{x}(x, t)=\text { mesh displacement } \tag{7.12.1c}
\end{equation*}
$$

such that they satisfy the following state and field equations:
Continuity Equation [Eq. (7.10.3b)]

$$
\begin{equation*}
\left.\frac{1}{B} \frac{\partial p}{\partial t}\right|_{\chi}+\frac{1}{B} c_{i} \frac{\partial p}{\partial x_{i}}+\frac{\partial v_{i}}{\partial x_{i}}=0 \tag{7.12.2a}
\end{equation*}
$$

Momentum Equation [Eq. (7.10.1b)]

$$
\begin{equation*}
\left.\rho \frac{\partial v_{i}}{\partial t}\right|_{\chi}+\rho c_{j} \frac{\partial v_{i}}{\partial x_{j}}=\frac{\partial \sigma_{i j}}{\partial x_{j}}+\rho g_{i} \tag{7.12.2b}
\end{equation*}
$$

## Free Surface Update Equation

We can employ either mesh rezoning equation [Eq. (7.11.3)]:

$$
\begin{equation*}
\left.\frac{\partial x_{i}}{\partial t}\right|_{\chi}+\left(\delta_{j k}-\alpha_{j k}\right) v_{k} \frac{\partial x_{i}}{\partial \chi_{j}}-v_{i}=0 \tag{7.12.2c}
\end{equation*}
$$

or the mesh updating equation [Eq. (7.11.17)]:

$$
\begin{equation*}
\left.\frac{\partial x_{i}}{\partial t}\right|_{\chi}-v_{i}-\sum_{\substack{j=1 \\ j \neq 1}}^{N S D} \frac{v_{j}-\hat{v}_{j}}{\hat{J}^{\underline{i}}} \hat{J}^{j \underline{i}}=-\frac{\hat{J}}{\hat{J}^{i \underline{i}}} w_{i} \tag{7.12.2d}
\end{equation*}
$$

The boundary conditions are as follows. It is required that:

$$
\begin{array}{ll}
v_{i}=\bar{v}_{i} & \text { on } \Gamma_{i}^{v} \\
\sigma_{i j} n_{j}=\bar{t}_{i} & \text { on } \Gamma_{i}^{t}
\end{array}
$$

where $b$ and $h$ are the prescribed boundary velocities and tractions, respectively; $n$ is the outward normal to $\Gamma_{i}^{v}$, and $\Gamma_{i}^{v}$ is the piecewise smooth boundary of the spatial domain, $\Omega$ and the decomposition of $\Gamma$ is given in Chapter 3.

## Constitutive Equation

$$
\begin{equation*}
\sigma_{i j}=-p \delta_{i j}+2 \mu D_{i j} \tag{7.12.5a}
\end{equation*}
$$

where

$$
\begin{equation*}
D_{i j}=\frac{1}{2}\left(\frac{\partial v_{i}}{\partial x_{j}}+\frac{\partial v_{j}}{\partial x_{i}}\right) \text { and } \mu=\mu\left(D_{i j}\right) \tag{7.12.5b}
\end{equation*}
$$

$\mu$ is the dynamic viscosity which is shear rate dependent. In Table 7.2 several Generalized Newtonian models (see e.g. Bird et al, 1977) are presented. The finite element method presented here is independent of the particular Generalized Newtonian model chosen.

### 7.13 Weak Form of Slightly Compressible Viscous Flow with Moving Boundary Problem

We denote the spaces of the test function and trial functions by:

$$
\begin{array}{ll}
\delta v_{i} \in \mathcal{U l}_{0}^{v} & \mathcal{U}_{0}^{v}=\left\{\delta v_{i} \mid \delta v_{i} \in C^{0}, \delta v_{i}=0 \text { on } \Gamma_{i}^{v}\right\} \\
v_{i} \in \mathcal{U}^{v} & \mathcal{U}^{v}=\left\{v_{i} \mid v_{i} \in C^{0}, v_{i}=\bar{v}_{i} \text { on } \Gamma_{i}^{v}\right\} \tag{7.13.1b}
\end{array}
$$

To establish the weak form of the momentum equation, Eq. (7.12.2b), we take the inner product of the momentum equation, Eq. (7.12.2b), with a test function $\delta v_{i}$ and integrate over the spatial region, that is:

$$
\begin{equation*}
\left.\int_{\Omega} \delta v_{i} \rho \frac{\partial v_{i}}{\partial t}\right|_{\chi} d \Omega+\int_{\Omega} \delta v_{i} \rho c_{j} \frac{\partial v_{i}}{\partial x_{j}} d \Omega-\int_{\Omega} \delta v_{i} \frac{\partial \sigma_{i j}}{\partial x_{j}} d \Omega-\int_{\Omega} \delta v_{i} \rho g_{i} d \Omega=0 \tag{7.13.2}
\end{equation*}
$$

It is noticed that the stresses in Eq. (7.13.2) are functions of the velocities. Applying integration by parts on the stress term yields:

$$
\begin{equation*}
\int_{\Omega} \delta v_{i} \frac{\partial \sigma_{i j}}{\partial x_{j}} d \Omega=\int_{\Omega} \frac{\partial}{\partial x_{j}}\left(\delta v_{i} \sigma_{i j}\right) d \Omega-\int_{\Omega} \frac{\partial\left(\delta v_{i}\right)}{\partial x_{j}} \sigma_{i j} d \Omega \tag{7.13.3}
\end{equation*}
$$

By the Gauss divergence theorem, the first term in the RHS of Eq. (7.13.3) can be written as a boundary integral, which is:

$$
\begin{equation*}
\int_{\Omega} \frac{\partial}{\partial x_{j}}\left(\delta v_{i} \sigma_{i j}\right) d \Omega=\int_{\Gamma^{t}} \delta v_{i}\left(n_{j} \sigma_{i j}\right) d \Gamma \tag{7.13.4}
\end{equation*}
$$

Substituting the specified boundary condition defined in Eq. (7.12.3b) into Eq. (7.13.4) gives:

$$
\begin{equation*}
\int_{\Gamma^{t}} \delta v_{i}\left(n_{j} \sigma_{i j}\right) d \Gamma=\int_{\Gamma^{t}} \delta v_{i} \bar{t}_{j} d \Gamma \tag{7.13.5}
\end{equation*}
$$

The second term in the RHS of Eq. (7.13.3) can be further expressed by using the definition of the constitutive equation, Eq. (7.12.5a). That gives:

$$
\begin{align*}
\int_{\Omega} \frac{\partial\left(\delta v_{i}\right)}{\partial x_{j}} \sigma_{i j} d \Omega & =\int_{\Omega} \frac{\partial\left(\delta v_{i}\right)}{\partial x_{j}}\left[-p \delta_{i j}+2 \mu D_{i j}\right] d \Omega \\
& =-\int_{\Omega} \frac{\partial\left(\delta v_{i}\right)}{\partial x_{i}} p d \Omega+\int_{\Omega} \frac{\partial\left(\delta v_{i}\right)}{\partial x_{i}} 2 \mu D_{i j} d \Omega \tag{7.13.6}
\end{align*}
$$

We now may use the decomposition of the velocity gradient in Eq. (7.13.6) into symmetric and antisymmetric parts:

$$
\begin{equation*}
\frac{\partial\left(\delta v_{i}\right)}{\partial x_{i}}=\delta L_{i j}=\delta D_{i j}+\delta \omega_{i j} \tag{7.13.7a}
\end{equation*}
$$

where

$$
\begin{equation*}
\delta D_{i j}=\frac{1}{2}\left[\frac{\partial\left(\delta v_{i}\right)}{\partial x_{j}}+\frac{\partial\left(\delta v_{j}\right)}{\partial x_{i}}\right\rfloor \text { and } \delta \omega_{i j}=\frac{1}{2}\left[\frac{\partial\left(\delta v_{i}\right)}{\partial x_{j}}-\frac{\partial\left(\delta v_{j}\right)}{\partial x_{i}}\right] \tag{7.13.7b}
\end{equation*}
$$

Since $\omega_{i j}$ is antisymmetric and $D_{i j}$ is symmetric, it leads to $\omega_{i j} D_{i j}=0$. Therefore, together with the constitutive equation, Eq. (7.12.5b), and Eq. (7.13.7a), we can express Eq. (7.13.6) as:

$$
\begin{equation*}
-\int_{\Omega} \frac{\partial\left(\delta v_{i}\right)}{\partial x_{i}} P d \Omega+\int_{\Omega} \frac{\mu}{2}\left[\frac{\partial\left(\delta v_{i}\right)}{\partial x_{j}}+\frac{\partial\left(\delta v_{j}\right)}{\partial x_{i}}\right]\left[\frac{\partial v_{i}}{\partial x_{j}}+\frac{\left.\partial v_{j}\right\rceil}{\partial x_{i}}\right] d \Omega \tag{7.13.8}
\end{equation*}
$$

Now, substitute Eqs. (7.13.5) and (7.13.8) into (7.13.2), the weak form for the momentum equation and associated boundary condition is obtained:

$$
\left.\int_{\Omega} \delta v_{i} \rho \frac{\partial v_{i}}{\partial t}\right|_{\chi} d \Omega+\int_{\Omega} \delta v_{i} \rho c_{j} \frac{\partial v_{i}}{\partial x_{j}} d \Omega
$$

$$
\begin{align*}
& -\int_{\Omega} \frac{\partial\left(\delta v_{i}\right)}{\partial x_{i}} P d \Omega+\int_{\Omega} \frac{\mu}{2} \frac{\partial\left(\delta v_{i}\right)}{\partial x_{j}}+\frac{\partial\left(\delta v_{j}\right)}{\partial x_{i}} \|\left[\frac{\partial v_{i}}{\partial x_{j}}+\frac{\left.\partial v_{j}\right\rceil}{\partial x_{i}}\right] d \Omega \\
& -\int_{\Omega} \delta v_{i} \rho g_{i} d \Omega-\int_{\Gamma^{t}} \delta v_{i} \bar{t}_{j} d \Gamma=0 \tag{7.13.9}
\end{align*}
$$

The weak forms for the continuity equation and the free surface update equation are simply obtained by taking the inner product with $\delta p$ and $\delta x_{i}$, respectively.

We may now state a suitable weak form for the momentum equation.

## Weak Form for Newtonian Fluid

Given density, $\rho$, bulk modules, $B$, and Cauchy stress function, $\sigma$, defined in Table 7.2, respectively, find $v \in \mathcal{U}^{v}, p \in \mathcal{U}^{p}$ and $x \in \mathcal{U}^{x}$ such that for every $\delta v \in \mathcal{U} u_{0}^{v}, \delta p \in \mathcal{U}_{0}^{p}$ and $\delta x \in \mathcal{U}_{0}^{x}:$

Continuity Equation [Eq. (7.12.2a)]

$$
\begin{equation*}
\left.\int_{\Omega} \delta p \frac{1}{B} \frac{\partial p}{\partial t}\right|_{\chi} d \Omega+\int_{\Omega} \delta p \frac{1}{B} c_{i} \frac{\partial p}{\partial x_{i}} d \Omega+\int_{\Omega} \delta p \frac{\partial v_{i}}{\partial x_{i}} d \Omega=0 \tag{7.13.10a}
\end{equation*}
$$

Momentum Equation [Eq. (7.12.2b)]

$$
\begin{align*}
& \left.\int_{\Omega} \delta v_{i} \rho \frac{\partial v_{i}}{\partial t}\right|_{\chi} d \Omega+\int_{\Omega} \delta v_{i} \rho c_{j} \frac{\partial v_{i}}{\partial x_{j}} d \Omega-\int_{\Omega} \frac{\partial\left(\delta v_{i}\right)}{\partial x_{i}} p d \Omega \\
& +\int_{\Omega} \frac{\mu}{2}\left[\frac{\partial\left(\delta v_{i}\right)}{\partial x_{j}}+\frac{\partial\left(\delta v_{j}\right)}{\partial x_{i}}\right]\left\lceil\frac{\partial v_{i}}{\partial x_{j}}+\frac{\partial v_{j}}{\partial x_{i}}\right] d \Omega-\int_{\Omega} \delta v_{i} \rho g_{i} d \Omega-\int_{\Gamma^{t}} \delta v_{i} \bar{t}_{j} d \Gamma=0 \tag{7.13.10b}
\end{align*}
$$

## Free Surface Update Equation

The mesh rezoning equation [Eq. (7.12.2c)]:

$$
\begin{equation*}
\left.\int_{\hat{\Omega}} \delta x_{i} \frac{\partial x_{i}}{\partial t}\right|_{\chi} d \hat{\Omega}+\int_{\hat{\Omega}}\left(\delta_{j k}-\alpha_{j k}\right) v_{k} \delta x_{i} \frac{\partial x_{i}}{\partial \chi_{j}} d \hat{\Omega}-\int_{\hat{\Omega}} \delta x_{i} v_{i} d \hat{\Omega}=0 \tag{7.13.10c}
\end{equation*}
$$

or the mesh updating equation [Eq. (7.12.2d)]:

$$
\begin{equation*}
\left.\int_{\hat{\Omega}} \delta x_{i} \frac{\partial x_{i}}{\partial t}\right|_{\chi} d \hat{\Omega}-\int_{\hat{\Omega}} \delta x_{i} \sum_{\substack{j=1 \\ j \neq 1}}^{N S D} \frac{v_{j}-\hat{v}_{j}}{\hat{J}^{i}} \hat{J}^{j i} d \hat{\Omega}-\int_{\hat{\Omega}} \delta x_{i}\left(v_{i}-\frac{\hat{J}}{\hat{J}^{i \underline{i}}} w_{i}\right) d \hat{\Omega}=0 \tag{7.13.10d}
\end{equation*}
$$

## Constitutive Equation

$$
\begin{equation*}
\sigma_{i j}=-p \delta_{i j}+2 \mu D_{i j} \tag{7.13.11e}
\end{equation*}
$$

Table 7.3 Weak Form of Slightly Compressible Viscous Flow with Moving Boundary Problem
7.13.1 Galerkin Approximation of Slightly Compressible Viscous Flow with Moving Boundaries
To obtain the semidiscrete equations by the Galerkin approximation, Eq. (7.13.10a) through (7.13.10d) is replaced by the finite functions $p^{h}, \mathbf{v}^{h}$ and $\mathbf{x}^{h}$. That is:

$$
\begin{equation*}
p \rightarrow p^{h}, \mathbf{x} \rightarrow \mathbf{x}^{h} \text { and } \mathbf{v} \rightarrow \mathbf{v}^{h} \tag{7.13.11}
\end{equation*}
$$

In particular, we wish to separate these functions into two parts, the unknown parts $w_{\mathbf{x}}^{h}$, $w_{p}^{h}$ and $w_{\mathbf{v}}^{h}$ and the prescribed boundary parts (essential boundary conditions) $\overline{\mathbf{x}}^{h}, \bar{p}^{h}$ and $\overline{\mathbf{v}}^{h}$, so that:

$$
\begin{align*}
& p^{h}=w_{p}^{h}+\bar{p}^{h}  \tag{7.13.12a}\\
& \mathbf{v}^{h}=w_{\mathbf{v}}^{h}+\overline{\mathbf{v}}^{h}  \tag{7.13.12b}\\
& \mathbf{x}^{h}=w_{\mathbf{x}}^{h}+\overline{\mathbf{x}}^{h} \tag{7.13.12c}
\end{align*}
$$

Given $\rho, \mathrm{B}, \sigma$ as before, find $v_{i}^{h}=w_{v_{i}}^{h}+\bar{v}_{i}^{h}, p^{h}=w_{p}^{h}+\bar{p}^{h}$ and $x_{i}^{h}=w_{x_{i}}^{h}+\bar{x}_{i}^{h}$, where $w_{v}^{h} \in \mathcal{U}_{0}^{v^{h}}, w_{p}^{h} \in \mathcal{U}_{0}^{p^{h}}$ and $w_{x}^{h} \in \mathcal{U}_{0}^{x^{h}}$, such that for every $\delta v^{h} \in \mathcal{U}_{0}^{v^{h}}, \delta p^{h} \in \mathcal{U}_{0}^{p^{h}}$ and $\delta x^{h} \in \mathcal{U}_{0}^{x^{h}}:$

## Galerkin Form of Continuity Equation

$$
\begin{align*}
& \left.\int_{\Omega} \delta p^{h} \frac{1}{B} \frac{\partial w_{p}^{h}}{\partial t}\right|_{\chi} d \Omega+\int_{\Omega} \delta p^{h} \frac{1}{B} c_{i} \frac{\partial w_{p}^{h}}{\partial x_{i}} d \Omega+\int_{\Omega} \delta p^{h} \frac{\partial v_{i}}{\partial x_{i}} d \Omega \\
& =-\left.\int_{\Omega} \delta p^{h} \frac{1}{B} \frac{\partial \bar{p}^{h}}{\partial t}\right|_{\chi} d \Omega-\int_{\Omega} \delta p^{h} \frac{1}{B} c_{i} \frac{\partial \bar{p}^{h}}{\partial x_{i}} d \Omega \tag{7.13.13a}
\end{align*}
$$

## Galerkin Form of Momentum Equation

$$
\left.\int_{\Omega} \delta v_{i}^{h} \rho \frac{\partial w_{v_{i}}^{h}}{\partial t}\right|_{\chi} d \Omega+\int_{\Omega} \delta v_{i}^{h} \rho c_{j} \frac{\partial w_{v_{i}}^{h}}{\partial x_{j}} d \Omega-\int_{\Omega} \frac{\partial\left(\delta v_{i}^{h}\right)}{\partial x_{i}} P d \Omega
$$

$$
\begin{align*}
& +\int_{\Omega} \frac{\mu}{2}\left[\frac{\partial\left(\delta v_{i}^{h}\right)}{\partial x_{j}}+\frac{\partial\left(\delta v_{j}^{h}\right)}{\partial x_{i}}\right]\left\lceil\left[\frac{\partial w_{v_{i}}^{h}}{\partial x_{j}}+\frac{\partial w_{v_{j}}^{h}}{\partial x_{i}}\right] d \Omega-\int_{\Omega} \delta v_{i}^{h} \rho g_{i} d \Omega\right. \\
& =\int_{\Gamma} \delta v_{i}^{h} h_{j} d \Gamma-\left.\int_{\Omega} \delta v_{i}^{h} \rho \frac{\partial \bar{v}_{i}^{h}}{\partial t}\right|_{\chi} d \Omega-\int_{\Omega} \delta v_{i}^{h} \rho c_{j} \frac{\partial \bar{v}_{i}^{h}}{\partial x_{j}} d \Omega \\
& -\int_{\Omega} \frac{\mu}{2}\left[\frac{\partial\left(\delta v_{i}^{h}\right)}{\partial x_{j}}+\frac{\partial\left(\delta v_{j}^{h}\right)}{\partial x_{i}}\right]\left\lceil\frac{\partial \bar{v}_{i}^{h}}{\partial x_{j}}+\frac{\partial \bar{v}_{j}^{h}}{\partial x_{i}}\right] d \Omega \tag{7.13.13b}
\end{align*}
$$

Galerkin Form of Free Surface Update Equation
The mesh rezoning equation:

$$
\begin{align*}
& \left.\int_{\hat{\Omega}} \delta x_{i}^{h} \frac{\partial w_{x_{i}}^{h}}{\partial t}\right|_{\chi} d \hat{\Omega}+\int_{\hat{\Omega}}\left(\delta_{j k}-\alpha_{j k}\right) v_{k} \delta x_{i}^{h} \frac{\partial w_{x_{i}}^{h}}{\partial \chi_{j}} d \hat{\Omega}-\int_{\hat{\Omega}} \delta x_{i}^{h} v_{i} d \hat{\Omega} \\
& =\left.\int_{\hat{\Omega}} \delta x_{i}^{h} \frac{\partial \bar{x}_{i}^{h}}{\partial t}\right|_{\chi} d \hat{\Omega}+\int_{\hat{\Omega}}\left(\delta_{j k}-\alpha_{j k}\right) v_{k} \delta x_{i}^{h} \frac{\partial \bar{x}_{i}^{h}}{\partial \chi_{j}} d \hat{\Omega} \tag{7.13.13c}
\end{align*}
$$

or the mesh updating equation:

$$
\begin{align*}
& \left.\int_{\hat{\Omega}} \delta x_{i}^{h} \frac{\partial w_{x_{i}}^{h}}{\partial t}\right|_{\chi} d \hat{\Omega}-\int_{\hat{\Omega}} \delta x_{i}^{h} \sum_{\substack{j=1 \\
j \neq 1}}^{N S D} \frac{v_{j}-\hat{v}_{j}}{\hat{J}^{i \underline{ }}} \hat{J}^{j i} d \hat{\Omega}-\int_{\hat{\Omega}} \delta x_{i}^{h}\left(v_{i}-\frac{\hat{J}}{\hat{J}^{\underline{i}}} w_{i}\right) d \hat{\Omega} \\
& =-\left.\int_{\hat{\Omega}} \delta x_{i}^{h} \frac{\partial \bar{x}_{i}^{h}}{\partial t}\right|_{\chi} d \hat{\Omega} \tag{7.13.13d}
\end{align*}
$$

7.13.2 Element Matrices for Slightly Compressible Viscous Flow with Moving Boundaries

The discrete forms of the continuity, momentum and mesh updating equations are presented next. First, we define:

$$
\begin{equation*}
v^{h}=w_{v}^{h}+\bar{v}^{h}=\sum_{A=1}^{N E Q v} N_{A}^{v}(\mathbf{x}) v_{A}(t)+\sum_{A=N E Q v+1}^{N U M N P v} N_{A}^{v}(\mathbf{x}) \bar{v}_{A}^{h}(t) \tag{7.13.14a}
\end{equation*}
$$

$$
\begin{align*}
& p^{h}=w_{p}^{h}+\bar{p}^{h}=\sum_{A=1}^{N E Q p} N_{A}^{p}(\mathbf{x}) p_{A}(t)+\sum_{A=N E Q p+1}^{N U M N P p} N_{A}^{p}(\mathbf{x}) \bar{p}_{A}^{h}(t)  \tag{7.13.14b}\\
& \hat{v}^{h}=w_{\hat{v}}^{h}+\overline{\hat{v}}^{h}=\sum_{A=1}^{N E Q \hat{v}} N_{A}^{\hat{v}}(\mathbf{x}) \hat{v}_{A}(t)+\sum_{A=N E Q \hat{v}+1}^{N U M N P \hat{v}} N_{A}^{\hat{v}}(\mathbf{x}) \overline{\hat{v}}_{A}^{h}(t)  \tag{7.13.14c}\\
& \delta \hat{v}^{h}=\sum_{A=1}^{N E Q \hat{v}} N_{A}^{\hat{v}}(\mathbf{X}) c_{A}^{\hat{v}}(t)  \tag{7.13.14d}\\
& \delta p^{h}=\sum_{A=1}^{N E Q p} N_{A}^{p}(\mathbf{X}) c_{A}^{p}(t)  \tag{7.13.14e}\\
& \delta v^{h}=\sum_{A=1}^{N E Q v} N_{A}^{v}(\mathbf{X}) c_{A}^{v}(t) \tag{7.13.14f}
\end{align*}
$$

where $N_{A}^{p}, N_{A}^{v}$ and $N_{A}^{\hat{v}}$ are the continuous element shape function for pressure, velocity and mesh velocity, respectively.
(ii) Mixed Formulation:

Without any loss of generality, the free surface is assumed perpendicular to the $\chi_{3}$ direction. The cofactors are

$$
\begin{align*}
& \hat{J}^{13}=\frac{\partial x_{2}}{\partial \chi_{1}} \frac{\partial x_{3}}{\partial \chi_{2}}-\frac{\partial x_{2}}{\partial \chi_{2}} \frac{\partial x_{3}}{\partial \chi_{1}}  \tag{7.13.20a}\\
& \hat{J}^{23}=\frac{\partial x_{1}}{\partial \chi_{2}} \frac{\partial x_{3}}{\partial \chi_{1}}-\frac{\partial x_{1}}{\partial \chi_{1}} \frac{\partial x_{3}}{\partial \chi_{2}}  \tag{7.13.20b}\\
& \hat{J}^{33}=\frac{\partial x_{1}}{\partial \chi_{2}} \frac{\partial x_{1}}{\partial \chi_{2}}-\frac{\partial x_{1}}{\partial \chi_{2}} \frac{\partial x_{2}}{\partial \chi_{1}} \tag{7.13.20c}
\end{align*}
$$

It should be noted that $x_{3}$ is the only unknown that defines the free surface which is assumed material (i.e. $w_{3}=0$ ).
(8d) Show that by substituting Eqs. (7.13.20) into Eq. (7.11.22b) yields:

$$
\left.\frac{\partial x_{3}}{\partial t}\right|_{\chi}+\frac{1}{\hat{J}^{33}}\left[\left(v_{1}-\hat{v}_{1}\right) \frac{\partial x_{2}}{\partial \chi_{2}}-\left(v_{2}-\hat{v}_{2}\right) \frac{\partial x_{1}}{\partial \chi_{2}}\right] \frac{\partial x_{3}}{\partial \chi_{1}}
$$

$$
\begin{equation*}
+\frac{1}{\hat{J}^{33}}\left[-\left(v_{1}-\hat{v}_{1}\right) \frac{\partial x_{2}}{\partial \chi_{1}}+\left(v_{2}-\hat{v}_{2}\right) \frac{\partial x_{1}}{\partial \chi_{1}}\right] \frac{\partial x_{3}}{\partial \chi_{2}}=0 \tag{7.13.21}
\end{equation*}
$$

(8e) Show that the convective term is:

$$
\begin{equation*}
\hat{L}_{A}=\int_{\Omega_{\chi}^{e}} \hat{N}_{A} \hat{c}_{m} \frac{\partial x_{i}}{\partial \chi_{m}} d \Omega_{\chi} \tag{7.13.22a}
\end{equation*}
$$

by defining:

$$
\begin{align*}
& \hat{c}_{1}=\frac{1}{\hat{J}^{33}}\left[\left(v_{1}-\hat{v}_{1}\right) \frac{\partial x_{2}}{\partial \chi_{2}}-\left(v_{2}-\hat{v}_{2}\right) \frac{\partial x_{1}}{\partial \chi_{2}}\right]  \tag{7.13.22b}\\
& \hat{c}_{2}=\frac{1}{\hat{J}^{33}}\left[-\left(v_{1}-\hat{v}_{1}\right) \frac{\partial x_{2}}{\partial \chi_{1}}+\left(v_{2}-\hat{v}_{2}\right) \frac{\partial x_{1}}{\partial \chi_{1}}\right]  \tag{7.13.22c}\\
& \hat{c}_{3}=0 \tag{7.13.22d}
\end{align*}
$$

### 7.15. Numerical Example

### 7.15.1 Elastic-plastic wave propagation problem

An elastic-plastic wave propagation problem is used to assess the ALE approach in conjunction with the regular fixed mesh method. The problem statement, given in Fig.1, represents a 1-D infinitely long, elastic-plastic hardening rod. Constant density and isothermal conditions are assumed to simplify the problem. Thus only the momentum equation and constitutive equation are considered for this problem. It should be noted that this elastic-plastic wave propagation problem does not require an ALE mesh and the problem was selected because it provides a severe test of the stress update procedure and because of the availability of an analytic solution. The problem is solved using 400 elements which are uniformly spaced with a mesh size of 0.1 . The mesh is arranged so that no reflected wave will occur during the time interval under consideration. Material properties and computational parameters are also depicted in Fig.7.16.1. Four stages are involved in this problem:
(1) $t \in\left[0, t_{1}\right]$, the mesh is fixed, and a square wave is generated at the origin;
(2) $t \in\left[t_{1}, t_{2}\right]$, the mesh is fixed and the wave travels along the bar;
(3) $t \in\left[t_{2}, t_{3}\right]$, two cases are studied:
case A: the mesh is moved uniformly to the left-hand side with a constant speed $-\hat{v}^{*}$; case B: same as Case A except the mesh is moved to the right;
(4) $t=t_{3}$, the stress is reported as a function of spatial coordinates in Figs. 7.16.2 and 7.16.3 for Case A and Case B, respectively.

For both cases, the momentum and stress transport are taken into account by employing the full upwind method for elastic and elastic-plastic materials. The results are compared to :
(1) Regular Galerkin method runs, in which all of the transport items are handled by the exact integration;
(2) Fixed mesh runs, in which the finite element mesh is fixed in space and the results are pretty close to the analytic solutions.

The results according to several time step size are reported in Table 7.16.1. The wave arrival time for both methods, with or without upwinding technique, agree well with the fixed mesh runs. However, the scheme without upwinding technique causes severe unrealistic spatial oscillations in Case A because of the significant transport effects. The new method proposed here eliminates these oscillations completely. Base on these studies, it is found that the transport of stresses as well as yield stress ( and back stresses if kinematic hardening) plays an important role in ALE computations for path-dependent materials, and the proposed update procedure is quite accurate and effective.

$$
\begin{aligned}
& \rho=1 \quad E=10^{4} \quad E / E_{T}=3 \sigma_{y 0}=75 \sigma_{0}=-100 \\
& \Delta x=\Delta \chi=0.1 \quad \hat{v}^{*}=0.25 \cdot \sqrt{E / \rho} \quad t_{1}=45 t_{2}=240 \quad t_{3}=320\left(\times 10^{-3}\right)
\end{aligned}
$$

1. $t \in\left[0, t_{1}\right]$ mesh fixed, wave generated


$$
x=0
$$

2. $t \in\left[t_{1}, t_{2}\right]$ mesh fixed, wavetravelling


$$
x=0
$$



3B. $t \in\left[t_{2}, t_{3}\right]$ Case B: mesh moving with $\hat{v}=+\hat{v}^{*}$


$$
x=0
$$

4. $t=t_{3} \quad$ report stress $v s$. spatial coordinate

Fig 7.16.1 Problem statement and computational parameters for wave propagation
Table 7.16.1 Time step sizes and numbers of time steps for elliptic-plastic wave propagation example:

| Time Step $(\Delta t)$ | $\Delta t / C r^{a}$ | Number of time steps |
| :---: | :---: | :---: |
| $0.040 \times 10^{-2}$ | 0.5 | 400 |
| $0.056 \times 10^{-2}$ | 0.7 | 286 |
| $0.072 \times 10^{-2}$ | 0.9 | 222 |

${ }^{a} C r=\Delta x / \sqrt{E / \rho}+|c|$

### 7.15.2 Breaking of a dam

This example is an attempt to model the breaking of a dam or more generally a flow with large free surface motion by the ALE formulations described in the previous sections. This problem, which has an approximate solution for an inviscid fluid flowing over a perfect frictionless bed, presents a formidable challenge when this solution is applied to mine tailings embankments. A detailed description of this problem can be found in Huerta \& Liu(1988).

The problem is solved without the restraints imposed by shallow water theory and only the case of flow over a still fluid (FSF) is considered. Study on another case of flow over a dry bed (FDB) can be found in the paper of Huerta \& Liu(1988). The accuracy of the ALE finite element approach is checked by solving the inviscid case, which has an analytical solution in shallow water theory; then, other viscous cases are studied and discussed.

Figure 7.8 shows a schematic representation of the flow over a still fluid The dimensionless problem is defined by employing the following characteristic dimensions: the length scale is the height of the dam, $H$, over the surface of the downstream still fluid; the characteristic velocity, $\sqrt{g H}$, is chosen to scale velocities; and $\rho g H$ is the pressure scale. The characteristic time is arbitrarily taken as the length scale over the velocity scale, i.e. $\sqrt{H / g}$. Consequently, if the fluid is Newtonian, the only dimensionless parameter associated with the field equations is the Reynolds number, $R_{e}=H \sqrt{g H} / v$, where $v$ is the kinematics viscosity. A complete parametric analysis may be found in Huerta (1987). Since the problem is studied in its dimensionless form, H is always set equal to one.

Along the upstream and downstream boundaries a frictionless condition is assumed, whereas on the bed perfect sliding is only imposed in the inviscid case (for viscous flows the velocities are set equal to zero). In the horizontal direction 41 elements of unit length are usually employed, while in the vertical direction one, three, five, or seven layers are taken. depending on the particular case (see Figure 7.9). For the inviscid analysis, $\Delta H=H=1$, as in Lohner et al (1984). In this problem both the Lagrange-Euler matrix method and the mixed formulation are equivalent because an Eulerian description is taken in the horizontal direction; in the vertical direction a Lagrangian description is used along the free surface while an Eulerian description is employed everywhere else.

Figure 7.10 compares the shallow water solution with the numerical results obtained by the one and three layers of elements meshes. Notice how the full integration of the Navier-Stokes equations smoothes the surface wave and slows down the initial motion of the flooding wave (recall that the Saint Venant equations predict a constant wave celerity, $\sqrt{g H}$, from $t=0$ ). No important differences exists between the two discretizations (i.e. one or three elements in depth); both present a smooth downstream surface and a clearly separate peak at the tip of the wave. It is believed that this peak is produced in large part by the sudden change in the vertical component of the particle velocity between still conditions and the arrival of the wave, instead of numerical oscillations only. Figure 7.11 shows the
difference between a Galerkin formulation of the rezoning equation, where numerical node to node oscillations are clear, and a Petrov-Galerkin integration of the free surface equation (i.e. the previous 41x3 element solution). The temporal criterion (Hughes and Tezduyar, 1984) is selected for the perturbation of the weighting functions, and, as expected (Hughes and Tezduyar, 1984; Hughes and Mallet, 1986), better results are obtained if the Courant number is equal to one. In the inviscid dam-break problem over a still fluid, the secondorder accurate Newmark scheme (Hughes and Liu, 1978) is used (i.e. $\gamma=0.5$ and $\beta=$ 0.25 ), while in all of the following cases numerical damping is necessary (i.e. $\gamma>0.5$ ) because of the small values of $\Delta H$; this numerical instability is discussed later

The computed free surfaces for different times and the previous Generalized Newtonian fluids are shown in Figures 7.14 and 7.15. It is important to point out that the results obtained with the Carreau-A model and $n=0.2$ are very similar to those of the Newtonian case with $R_{e}=300$, whereas for the Bingham material with $\mu_{p}=1 \times 10^{2} P_{a} \bullet s$ the free surface shapes resemble more closely those associated with $R_{e}=3000$; this is expected because the range of shear rate for this problem is from 0 up to 20-30 $s^{-1}$ It should also be noticed that both Bingham cases present larger oscillations at the free surface and that even for the $\mu_{p}=1 \times 10^{3} P_{a} \bullet s$ case the flooding wave moves faster than that for the Carreau models. Two main reasons can explain such behavior; first, unless uneconomical time-steps are chosen, oscillations appear in the areas where the fluid is at rest because of the extremely high initial viscosity $\left(1000 \mu_{p}\right)$; second, the larger shear rates occur at the tip of the wave, and it is in this area that the viscosity suddenly drops at least two orders of magnitude, creating numerical oscillations.

## Exercise 7.1

Observe that if the Jacobian described in Eq. (7.4.3a) is:

$$
\begin{equation*}
J=\operatorname{det}\left(\frac{\partial \mathbf{x}}{\partial \mathbf{X}}\right)=\varepsilon_{i j k} \frac{\partial x_{i}}{\partial X_{1}} \frac{\partial x_{j}}{\partial X_{2}} \frac{\partial x_{k}}{\partial X_{3}} \tag{7.4.11a}
\end{equation*}
$$

where $\varepsilon_{i j k}$ is the permutation symbol, then $\dot{J}$ becomes:

$$
\begin{equation*}
\dot{J}=\frac{\partial\left(v_{1}, x_{2}, x_{3}\right)}{\partial\left(X_{1}, X_{2}, X_{3}\right)}+\frac{\partial\left(x_{1}, v_{2}, x_{3}\right)}{\partial\left(X_{1}, X_{2}, X_{3}\right)}+\frac{\partial\left(x_{1}, x_{2}, v_{3}\right)}{\partial\left(X_{1}, X_{2}, X_{3}\right)} \tag{7.4.11b}
\end{equation*}
$$

where

$$
\frac{\partial(a, b, c)}{\partial\left(X_{1}, X_{2}, X_{3}\right)}=\left|\begin{array}{ccc}
\frac{\partial a}{\partial X_{1}} & \frac{\partial a}{\partial X_{2}} & \frac{\partial a}{\partial X_{3}}  \tag{7.4.11c}\\
\frac{\partial b}{\partial X_{1}} & \frac{\partial b}{\partial X_{2}} & \frac{\partial b}{\partial X_{3}} \\
\frac{\partial c}{\partial X_{1}} & \frac{\partial c}{\partial X_{2}} & \frac{\partial c}{\partial X_{3}}
\end{array}\right|
$$

for arbitrary scalars $a, b$, and $c$, and $v_{i}=\dot{x}_{i}$.
Using the chain rule on $\frac{\partial v_{1}}{\partial X_{j}}$, show that:

$$
\begin{equation*}
\frac{\partial\left(v_{1}, v_{2}, v_{3}\right)}{\partial\left(X_{1}, X_{2}, X_{3}\right)}=\sum_{m=1}^{3} \frac{\partial v_{1}}{\partial x_{m}} \frac{\partial\left(x_{m}, x_{2}, v_{3}\right)}{\partial\left(X_{1}, X_{2}, X_{3}\right)}=\frac{\partial v_{1}}{\partial x_{1}} J \tag{7.4.12a}
\end{equation*}
$$

Similarly, show that:

$$
\begin{equation*}
\dot{J}=J v_{k, k} \tag{7.4.12b}
\end{equation*}
$$

Exercise 7.2 Updated ALE Conservation of Angular Momentum
The principle of conservation of angular momentum states that the time rate of change of the angular momentum of a given mass with respect to a given point, say the origin of the reference frame, is equal to the applied torque referred to the same point. That is:

$$
\begin{equation*}
\frac{D}{D t} \int_{\Omega} \mathbf{x} \times \rho(\mathbf{x}, t) \mathbf{v}(\mathbf{x}, t) d \Omega=\int_{\Omega} \mathbf{x} \times \mathbf{b}(\mathbf{x}, t) d \Omega+\int_{\Gamma} \mathbf{x} \times \mathbf{t}(\mathbf{x}, t) d \Gamma \tag{7.4.13a}
\end{equation*}
$$

It should be noticed that the left hand side of Eq. (7.4.13a) is simply $\dot{\mathbf{H}}$.
(2a) Show that:

$$
\begin{align*}
\dot{\mathbf{H}} & =\int_{\Omega_{0}} \frac{D}{D t}(\mathbf{x}) \times(\rho \mathbf{v}) J d \Omega_{0}+\int_{\Omega_{0}} \mathbf{x} \times \frac{D}{D t}(\rho \mathbf{v} J) d \Omega_{0}  \tag{7.4.13b}\\
& =\int_{\Omega} \mathbf{x} \times\left[\frac{D}{D t}(\rho \mathbf{v})+(\rho \mathbf{v}) \operatorname{div}(\mathbf{v})\right] d \Omega
\end{align*}
$$

Hint, in deriving the above equation, the following pieces of information have been used of (1) $\mathbf{x}_{, t[\mathbf{X}]}=\mathbf{v}$; (2) $\mathbf{v} \times(\rho \mathbf{v})=\mathbf{0}$ and (3) Eq. (7.3.6b).

Now, show that substituting Eqs. (7.4.13b) into Eq. (7.4.13a) yields:

$$
\begin{align*}
& \int_{\Omega} \mathbf{x} \times\left[\frac{D}{D t}(\rho \mathbf{v})+(\rho \mathbf{v}) \operatorname{div}(\mathbf{v})\right] d \Omega  \tag{7.4.14}\\
& =\int_{\Omega} \mathbf{x} \times \mathbf{b}(\mathbf{x}, t) d \Omega+\int_{\Gamma} \mathbf{x} \times(\mathbf{n} \cdot \sigma) d \Gamma
\end{align*}
$$

(2b) Show that by employing the divergence theorem and the momentum equations given in Eq. (7.4.6), the component form of Eq. (7.4.14) is:

$$
\begin{equation*}
\int_{\Omega} \varepsilon_{i j k} \sigma_{j k} d \Omega=0 \tag{7.4.15}
\end{equation*}
$$

(2c) If the Cauchy stress tensor, $\sigma$, is smooth within $\Omega$, then the conservation of angular momentum leads to the symmetry condition of the Cauchy (true) stress via Eq. (7.4.15) and is given as:

$$
\begin{equation*}
\sigma_{i j}=\sigma_{j i} \tag{7.4.16}
\end{equation*}
$$

Exercise 7.3 Updated ALE Conservation of Energy
Energy conservation is expressed as (see chapter 3):

$$
\begin{equation*}
\frac{D}{D t} \int_{\Omega} \rho E d \Omega=\int_{\Gamma} \sigma_{j i} n_{j} v_{i} d \Gamma+\int_{\Omega} \rho b_{i} v_{i} d \Omega-\int_{\Gamma} q_{i} n_{i} d \Gamma+\int_{\Omega} \rho s d \Omega \tag{7.4.17}
\end{equation*}
$$

where $q_{i}$ is the heat flux leaving the boundary $\partial \Omega_{x}$. Recall that $E$ is the specific total energy density and is related to the specific internal energy $e$, by:

$$
\begin{equation*}
E=e+\frac{V^{2}}{2} \tag{7.4.18a}
\end{equation*}
$$

where $e=e(\theta, \rho)$ with $\theta$ being the thermodynamic temperature and $\rho s$ is the specific heat source, i.e. the heat source per unit spatial volume and $V^{2}=v_{i} v_{i}$. The Fourier law of heat conduction is:

$$
\begin{equation*}
q_{i}=-k_{i j} \theta_{, j} \tag{7.4.18b}
\end{equation*}
$$

(3a) Show that the energy equation is (hint, use integration by parts and the divergence theorem):

$$
\begin{equation*}
(\rho E)_{, t[\chi]}+\left(\rho E c_{j}\right)_{, j}+\rho E \hat{v}_{j, j}=\left(\sigma_{i j} v_{i}\right)_{, j}+b_{j} v_{j}+\left(k_{i j} \theta_{, j}\right)_{, i}+\rho s \tag{7.4.19a}
\end{equation*}
$$

(3b) If there is sufficient smoothness, time differentiate Eq. (7.4.19a) via the chain rule and make use of the continuity equation to show that Eq. (7.4.19a) reduces to:

$$
\begin{equation*}
\rho\left\{E_{, t[\chi]}+E_{, j} c_{j}\right\}=\left(\sigma_{i j} v_{i}\right)_{, j}+b_{j} v_{j}+\left(k_{i j} \theta_{, j}\right)_{, i}+\rho s \tag{7.4.19b}
\end{equation*}
$$

or, in index free notation:

$$
\begin{equation*}
\rho\left\{E_{, t[\chi]}+\mathbf{c} \cdot \operatorname{grad} E\right\}=\operatorname{div}(\mathbf{v} \cdot \sigma)+\mathbf{v} \cdot \mathbf{b}+\operatorname{div}(\mathbf{k} \cdot \operatorname{grad} \theta)+\rho s \tag{7.4.19c}
\end{equation*}
$$

(3c) Show that the above equations can be specified in the Lagrangian description by choosing:

$$
\begin{equation*}
\chi=\mathbf{X} ; \quad \hat{\phi}=\phi ; \quad \mathbf{c}=\mathbf{0} ; \quad J=\operatorname{det}\left(\frac{\partial \mathbf{x}}{\partial \mathbf{X}}\right) \tag{7.4.20a}
\end{equation*}
$$

and they are given by:

$$
\begin{equation*}
\rho E_{, t[\chi]}=\left(\sigma_{i j} v_{i}\right)_{, j}+b_{j} v_{j}+\left(k_{i j} \theta_{, j}\right)_{, i}+\rho s \tag{7.4.20b}
\end{equation*}
$$

or, in index free notation:

$$
\begin{equation*}
\rho E_{, t[\chi]}=\operatorname{div}(\mathbf{v} \cdot \sigma)+\mathbf{v} \cdot \mathbf{b}+\operatorname{div}(\mathbf{k} \cdot \operatorname{grad} \theta)+\rho s \tag{7.4.20c}
\end{equation*}
$$

(3d) Similarly, show that the Eulerian energy equation is obtained by choosing:

$$
\begin{equation*}
\chi=\mathbf{x} ; \quad \hat{\phi}=\mathbf{1} ; \quad \mathbf{c}=\mathbf{v} ; \quad \hat{\mathbf{v}}=\mathbf{0} ; \quad J=\operatorname{det}\left(\frac{\partial \mathbf{x}}{\partial \mathbf{X}}\right)=1 \tag{7.4.21a}
\end{equation*}
$$

and they are given by:

$$
\begin{equation*}
\rho\left\{E_{, t[\chi]}+E_{, j} v_{j}\right\}=\left(\sigma_{i j} v_{i}\right)_{, j}+b_{j} v_{j}+\left(k_{i j} \theta_{, j}\right)_{, i}+\rho s \tag{7.4.21b}
\end{equation*}
$$

or

$$
\begin{equation*}
\rho\left\{E_{, t[\chi]}+\mathbf{v} \cdot \operatorname{grad} E\right\}=\operatorname{div}(\mathbf{v} \cdot \sigma)+\mathbf{v} \cdot \mathbf{b}+\operatorname{div}(\mathbf{k} \cdot \operatorname{grad} \theta)+\rho s \tag{7.4.21c}
\end{equation*}
$$

## Exercise 7.4

Show Eqs (7.13.10a), (7.13.10c), and (7.13.10d).
Exercise 7.5 Galerkin Approximation
Show the following Galerkin approximation by substituting these approximation functions, Eqs (7.13.12), into Eqs. (7.13.10).

Exercise 7.6 The Continuity Equation
(6a) Show that:

$$
\begin{equation*}
\mathbf{M}^{p} \dot{\mathbf{P}}+\mathbf{L}^{p}(\mathbf{P})+\mathbf{G}^{T} \mathbf{v}=f^{e x t p} \tag{7.13.15a}
\end{equation*}
$$

where $\mathbf{M}^{p}$ is the generalized mass matrices for pressure; $\mathbf{L}^{p}$ is the generalized convective terms for pressure; $\mathbf{G}$ is the divergence operator matrix; $f^{e x t p}$ is the external load vector; $\mathbf{P}$ and $\mathbf{v}$ are the vectors of unknown nodal values for pressure and velocity, respectively; and $\dot{\mathbf{P}}$ is the time derivative of the pressure.
(6b) Show that:

$$
\begin{align*}
& M_{A B}^{P}=\int_{\Omega^{e}} \frac{1}{B} N_{A}^{p} N_{B}^{p} d \Omega  \tag{7.13.15b}\\
& L_{A}^{P}=\int_{\Omega^{e}} \frac{1}{B} N_{A}^{p} c_{k} \frac{\partial p}{\partial x_{k}} d \Omega  \tag{7.13.15c}\\
& G_{A B}^{P}=\int_{\Omega^{e}} N_{A}^{p} \frac{\partial N_{B}}{\partial x_{m}} d \Omega \tag{7.13.15d}
\end{align*}
$$

Example 7.2 1D Advection-Diffusion Equation

$$
\begin{aligned}
& 2 P_{e} \phi_{, x}-\phi_{, x x}=0 \\
& P_{e}=1.5 \quad \tau=0.438 \quad \Delta x=1
\end{aligned}
$$

Exercise 7.7 The Momentum Equation
(7a) Show that:

$$
\begin{equation*}
\mathbf{M a}+\mathbf{L}(\mathbf{v})+\mathbf{K}_{\mu} \mathbf{v}-\mathbf{G} \mathbf{P}=f^{e x t v} \tag{7.13.16a}
\end{equation*}
$$

where $\mathbf{M}$ is the generalized mass matrices for velocity; $\mathbf{L}$ is the generalized convective terms for velocity; $\mathbf{G}$ is the divergence operator matrix; $f^{\text {extv }}$ is the external load vector applied on the fluid; $\mathbf{K}_{\mu}$ is the fluid viscosity matrix; $\mathbf{P}$ and $\mathbf{v}$ are the vectors of unknown nodal values for pressure and velocity, respectively; and $\mathbf{P}$ and a are the time derivative of the pressure, and the material velocity, holding the reference fixed.
(7b) Show that:

$$
\begin{align*}
& M_{A B}=\int_{\Omega^{e}} \rho N_{A} N_{B} d \Omega  \tag{7.13.16b}\\
& L_{A}=\int_{\Omega^{e}} \rho N_{A} c_{m} \frac{\partial v_{i}}{\partial x_{m}} d \Omega \tag{7.13.16c}
\end{align*}
$$

$$
\begin{equation*}
\mathbf{K}_{\mu}=\int_{\Omega^{e}} \mathbf{B}^{T} \mathbf{D B} d \Omega \tag{7.13.16d}
\end{equation*}
$$

where

$$
\begin{align*}
& \mathbf{B}=\left[\begin{array}{l}
\left.\mathbf{B}_{1} \cdots \mathbf{B}_{a} \cdots \mathbf{B}_{\mathrm{NEN}}\right] \\
\left.\mathbf{B}_{a}^{T}=\left\lvert\, \begin{array}{cccccc}
\frac{\partial N_{a}}{\partial x_{1}} & \frac{\partial N_{a}}{\partial x_{2}} & 0 & 0 & 0 & \frac{\partial N_{a}}{\partial x_{3}} \\
0 & \frac{\partial N_{a}}{\partial x_{1}} & \frac{\partial N_{a}}{\partial x_{2}} & 0 & \frac{\partial N_{a}}{\partial x_{3}} & 0 \\
0 & 0 & 0 & \frac{\partial N_{a}}{\partial x_{3}} & \frac{\partial N_{a}}{\partial x_{2}} & \frac{\partial N_{a}}{\partial x_{1}}
\end{array}\right.\right] \\
\mathbf{D}=\left|\begin{array}{ccccc}
2 \mu & 0 & 0 & 0 & 0 \\
0 & \mu & 0 & 0 & 0 \\
0 & 0 & 2 \mu & 0 & 0 \\
0 \\
0 & 0 & 0 & 2 \mu & 0 \\
0 & 0 & 0 & 0 & \mu
\end{array}\right| \\
0
\end{array} 0\right.  \tag{7.13.17a}\\
& 0
\end{align*} 0
$$

## Exercise 7.8 The Mesh Updating Equation

(8a) Show that:

$$
\begin{equation*}
\hat{\mathbf{M}} \hat{\mathbf{v}}+\hat{\mathbf{L}}(\mathbf{x})-\hat{\mathbf{M}} \mathbf{v}=f^{e x t x} \tag{7.13.18a}
\end{equation*}
$$

where $\hat{\mathbf{M}}$ is the generalized mass matrices for mesh velocity; $\hat{\mathbf{L}}$ is the generalized convective terms for mesh velocity; $f^{\text {extx }}$ is the external load vector; and $\hat{\mathbf{v}}$ is the vectors of unknown nodal values for mesh velocity.
(8b) Show that:

$$
\begin{equation*}
\hat{M}_{A B}=\int_{\hat{\Omega}^{e}} \rho \hat{N}_{A} \hat{N}_{B} d \hat{\Omega} \tag{7.13.18b}
\end{equation*}
$$

The convective term is defined as follows:
(i) Lagrangian-Eulerian Matrix Method:

Define:

$$
\begin{equation*}
\hat{c}_{i}=\left(\delta_{i j}-\alpha_{i j}\right) v_{j} \tag{7.13.19a}
\end{equation*}
$$

(8c) Show that the convective term is:

$$
\begin{equation*}
\hat{L}_{A}=\int_{\hat{\Omega}^{e}} \hat{N}_{A} \hat{c}_{m} \frac{\partial x_{i}}{\partial \chi_{m}} d \hat{\Omega} \tag{7.13.19b}
\end{equation*}
$$

## Exercise 6

Replacing the test function $\delta v_{i}$ by $\delta v_{i}+\tau \rho c_{j} \frac{\delta v_{i}}{\delta x_{j}}$, show that the streamline-upwind/Petrov-Galerkin formulation for the momentum equation is:

$$
\begin{aligned}
& 0=\left.\int_{\Omega} \delta v_{i} \rho \frac{\partial v_{i}}{\partial t}\right|_{\chi} d \Omega+\int_{\Omega} \delta v_{i} \rho c_{j} \frac{\partial v_{i}}{\partial x_{j}} d \Omega-\int_{\Omega} \frac{\partial\left(\delta v_{i}\right)}{\partial x_{i}} P d \Omega-\int_{\Omega_{x}} \delta v_{i} \rho g_{i} d \Omega \\
& +\int_{\Omega} \frac{\mu}{2}\left[\frac{\partial\left(\delta v_{i}\right)}{\partial x_{j}}+\frac{\partial\left(\delta v_{j}\right)}{\partial x_{i}}\right]\left\lceil\frac{\partial v_{i}}{\partial x_{j}}+\frac{\left.\partial v_{j}\right\rceil}{\partial x_{i}}\right] d \Omega-\int_{\Gamma} \delta v_{i} h_{j} d \Gamma \quad \Leftarrow \text { Galerkin } \\
& \left.+\sum_{e=1}^{N U M E L} \int_{\Omega^{e}} \tau \rho c_{j} \frac{\delta v_{i}}{\delta x_{j}}\left|\rho \frac{\partial v_{i}}{\partial t}\right|_{\chi}+\rho c_{j} \frac{\partial v_{i}}{\partial x_{j}}-\frac{\partial \sigma_{i j}}{\partial x_{j}}-\rho g_{i}\right\rceil d \Omega
\end{aligned} \Leftarrow \text { StreamlineUpwind } \quad l
$$

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---------- Backup of the previous version
In section 7.1, a brief introduction of the ALE is given. In section 7.2, the kinematics in ALE formulation is described. In section 7.3, the Lagrangian versus referential updates is given. In section 7.4, the updated ALE balance laws in referential description is described. In section 7.5, the strong form of updated ALE conservation laws in referenctial description is derived. In section 7.6, an example of dam-break is used to show the application of updated ALE. In section 7.7, the updated ALE is applied to the pathdependent materials extensively where the strong form, the weak form and the finite element decretization are derived. In this section, emphasize is focused on the stress update procedure. Formulations for regular Galerkin method, Streamline-upwind/PetrovGalerkin(SUPG) method and operator splitting method are derived respectively. All the path-dependent state variables are updated with a similar procedure. In addition, the stress update procedures in 1D case are specified with the elastic and elastic-plastic wave propagation examples to demonstrate the effectiveness of the ALE method. In section 7.8, the total ALE method, the counterpart of updated ALE method, is studied.

### 7.1 Introduction

The theory of continuum mechanics (Malvern [1969], Oden [1972]) serves to establish an idealization and a mathematical formulation for the physical responses of a material body which is subjected to a variety of external conditions such as thermal and mechanical loads. Since a material body $B$ defined as a continuum is a collection of material particles $p$, the purpose of continuum mechanics is to provide governing equations which describe the deformations and motions of a continuum in space and time under thermal and mechanical disturbances.

The mathematical model is achieved by labelling the points in the material body $B$ by the real number planes $\Omega$, where $\Omega$ is the region (or domain) of the Euclidean space. Henceforth, the material body B is replaced by an idealized mathematical body, namely, the region $\Omega$. Instead of being interested in the atomistic view of the particles $p$, the description of the behavior of the body $B$ will only pertain to the regions of Euclidean space .

Equations describing the behavior of a continuum can generally be divided into four major categories: (1) kinematic, (2) kinetic (balance laws), (3) thermodynamic, and (4) constitutive. Detailed treatments of these subjects can be found in many standard texts.

The two classical descriptions of motion, are the Lagrangian and Eulerian descriptions. Neither is adequate for many engineering problems involving finite deformation especially when using finite element methods. Typical examples of these are fluid-structure-solid interaction problems, free-surface flow and moving boundary problems, metal forming processes and penetration mechanics, among others.

Therefore, one of the important ingredients in the development of finite element methods for nonlinear mechanics involves the choice of a suitable kinematic description for each particular problem. In solid mechanics, the Lagrangian description is employed extensively for finite deformation and finite rotation analyses. In this description, the calculations follow the motion of the material and the finite element mesh coincides with the
same set of material points throughout the computation. Consequently, there is no material motion relative to the convected mesh. This method has its popularity because
(1) the governing equations are simple due to the absence of convective effects, and
(2) the material properties, boundary conditions, stress and strain states can be accurately defined since the material points coincide with finite element mesh and quadrature points throughout the deformation. However, when large distortions occur, there are disadvantages such as:
(1) the meshes become entangled and the resulting shapes may yield negative volumes, and
(2) the time step size is progressively reduced for explicit time-stepping calculations.

On the other hand, the Eulerian description is preferred when it is convenient to model a fixed region in space for situations which may involve large flows, large distortions, and mixing of materials. However, convective effects arise because of the relative motion between the flow of material and the fixed mesh, and these introduce numerical difficulties. Furthermore, the material interfaces and boundaries may move through the mesh which requires special attention.

In this chapter, a general theory of the Arbitrary Lagrangian-Eulerian (ALE) description is derived. The theory can be used to develop an Eulerian description also. The definitions of convective velocity and referential or mesh time derivatives are given. The balance laws, such as conservation of mass, balances of linear and angular momentum and conservation of energy are derived within the mixed Lagrangian-Eulerian concept. The degenerations of the mixed description to the two classical descriptions, Lagrangian and Eulerian, are emphasized. The formal statement of the initial/boundary-value problem for the ALE description is also discussed.

### 7.2 Kinematics in ALE formulation

### 7.2.1 Mesh Displacement, Mesh Velocity and Mesh Acceleration

In order to complete the referential description, it is necessary to define the referential motion; this motion is called the mesh motion in the finite element formulation.

The motion of the body $B$, which occupies a reference region $\Omega_{\chi}$, is given by

$$
\begin{equation*}
\mathbf{x}=\hat{\phi}(\chi, t)=\chi+\hat{\mathbf{u}}(\chi, t)=\phi(\mathbf{X}, t) \tag{7.2.7}
\end{equation*}
$$

This ALE referential(mesh) region $\Omega_{\chi}$ is specified throughout and its motion is defined by the mapping function $\hat{\phi}$ such that the motion of $\chi \in \Omega_{\chi}$ at time $t$ is denoted by $\chi \in \Omega_{\chi}$ and $\hat{\mathbf{u}}(\chi, t)$ is the mesh displacement in the finite element formulation. It is noted that even thought in general the mesh function $\hat{\phi}$ is different from the material function $\phi$, the two motions are the same as given in Eq.(7.2.7). The corresponding velocity (mesh velocity) and acceleration (mesh acceleration) are defined as :

$$
\begin{equation*}
\hat{\mathbf{v}}=\left.\frac{\partial \mathbf{x}}{\partial t}\right|_{[\chi]}=\mathbf{x}_{, t[\chi]}=\hat{\mathbf{u}}_{, t[\chi]} \quad \text { mesh velocity } \tag{7.2.8a}
\end{equation*}
$$

and

$$
\begin{equation*}
\hat{\mathbf{a}}=\left.\frac{\partial \hat{\mathbf{v}}}{\partial t}\right|_{[\chi]}=\hat{\mathbf{v}}_{, t[\chi]} \quad \text { mesh acceleration } \tag{7.2.8b}
\end{equation*}
$$

The motion $\hat{\phi}$ is arbitrary and the usefulness of the referential descripation will depend on how this motion is chosen.

Depending on the choice of $\chi$, we can obtain the Lagranginan description by setting $\chi=\mathbf{X}$ and $\hat{\phi}=\phi$, the Eulerian description by setting $\chi=\mathbf{x}$, and the ALE description by setting $\hat{\phi} \neq \phi$. The general referential description is referred to as Arbitrary Lagrangian-Eulerian (ALE) in the finite element formulation. In this description, the function $\hat{\phi}$ must be specified such that the mapping between $\mathbf{x}$ and $\chi$ is one to one. With this assumption and by the composition of the mapping (denoted by a circle), a third mapping is defined such that

$$
\begin{equation*}
\chi=\psi(\mathbf{X}, t)=\hat{\phi}^{-1} \circ \phi(\mathbf{X}, t) \tag{7.2.10}
\end{equation*}
$$

Similarly, for this motion displacement, velocity and acceleration variables can be defined.

However, this is not necessary. These displacement, velocity and acceleration variables can instead be defined with the aid of the chain rule and the appropriate mappings.

The schematic set up of these descriptions in one-dimension is shown in Fig. 1, and a summary of the three descriptions is given in Table 7.1.

Fig.7.2 is shown to compare the three descriptions further, where the 1 D motion of the material is specified as:

$$
x=\left(1-X^{2}\right) t+X t^{2}+X
$$

Fig 7.2 Comparsion of Lagranian, Eulerian, ALE description

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